Hierarchy of Maximal ideals in the Banach Algebra

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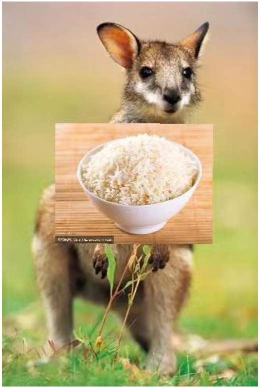
Introduction

The recent development of set theory and applications of set theory in diverse fields of mathematics such as Banach spaces, operator algebras, topology often involves elements of mathematical logic in the form of forcing since many results in this field are undecidable. It also often reduces to uncountable combinatorial arguments.
– Piotr Koszmider.



C*-algebra I

X is linear space with respect to addition and scalar multiplication over $\mathbb{C}$, where $\|\cdot\|$ is the norm defined on X, that is to say, X is an abelian group with respect to addition, which satisfies

1° Distribution $(\alpha+\beta)x = \alpha x + \beta x$, $\gamma(x+y) = \gamma x + \gamma y$;

2° Association $(\alpha\beta)x = \alpha(\beta x)$;

3° $1x = x$.

and $\|\cdot\|$ is a unary function on X, which satisfies:

i) $\|x\| \geq 0$, $\|x\| = 0$ if and only if $x = 0$;
ii) $\|\alpha x\| = |\alpha| \, \|x\|$;
iii) $\|x+y\| \leq \|x\| + \|y\|$. We assume the norm is complete.

The multiplication $\cdot$ is a binary operation on X, which satisfies

1. Association $(xy)z = x(yz)$;
2. Commutation $xy = yx$;

C*-algebra II

3. $\alpha xy = x\alpha y = xy\alpha$;
4. $x(y+z) = xy + xz$.

Also, $\cdot$ is compatible to the norm $\|\cdot\|$

iv) $\|xy\| \leq \|x\|\|y\|$;

We further assume X is unital, i.e. X has an identity element e. X which satisfies the axiom 1°-3°, i)-iv) and 1-4 is said to be a Banach algebra. The involution $*: X \to X$ is an anti-scalar linear transformation on X, which satisfies:

I. $(\alpha x)^* = \bar{\alpha} x^*$.
II. $(x+y)^* = x^* + y^*$.
III. $(xy)^* = y^* x^*$.
IV. $(x^*)^* = x$

Banach algebra with an involution is said to be a *-algebra.

C*-algebra III

If further *-operation is compatible to the norm $\|\cdot\|$, i.e.

v) $\|x^*x\| = \|x^*\|\|x\|$ and

vi) $\|x^*\| = \|x\|$ (wiki has typos here!)

or equivalently,

v′) $\|xx^*\| = \|x\|^2$.

The Banach algebra is said to be a C*-algebra.
Evidently, there are many seemingly superfluous assumptions like

- commutativity 2,
- compatible with norm iv), v′) and
- unital $e \in X$.

One main objective of this article is to study the situation when these assumptions are dropped out or ask whether they are consistent with ZFC.

ad hoc

Lemma 1

The maximal ideal m *of a Banach algebra* X *is closed in* X*.*

Lemma 2

If a Banach algebra X *is separable, it contains a dense subalgebra.*

Theorem 3 (Gelfand-Mazur)

Every Banach division algebra (particularly, X/m *when* X *is a commutative unital ring) is isomorphic to the complex number field* $\mathbb{C}$*.*

Proof.

It suffices to prove for any $a \in X$, there exists $z \in \mathbb{C}$ such that $a = ze$. Expand $(ze - a)^{-1}$. $\square$

Multiplicative functional I

A linear functional f on X is said to be multiplicative if

$$f(xy) = f(x)f(y), \quad x, y \in X,$$

and sub-multiplicative if $=$ is replaced by $\leq$.

Theorem 4 (Gleason, Kahane, Zelazko)

If f is a linear functional on a Banach algebra X, such that $f(e) = 1$ and $f(x) \neq 0$ for every invertible $x \in X$, the f multiplicative.

Proof.

Rudin, p.251-252. [Hint: $a^2 \in \ker f$ if $a \in \ker f$.] $\square$

Theorem 5 (Kaplansky)

Let A be a C-algebra. A is commutative if and only if the nilradical, i.e. the set all nilpotent elements of A (it's an ideal when 2 is satisfied) is 0.*

Multiplicative functional II

Lemma 6

Let A be a noncommutative C-algebra. Then there is a nonzero $x \in A$ with $x^2 = 0$, or equivalently, $x^*x \perp xx^*$.(Mathoverflow 435659.)*

Theorem 7

The following five statements about C-algebra A is equivalent.*

1. *A is commutative;*
2. *Every hereditary subalgebra of A is a closed ideal;*
3. *Every left(right) closed ideal is a closed ideal;*
4. *Every left(right) closed ideal is a closed under involution;*
5. *Every pure state is multiplicative linear functional;*
6. *For any positive element a, $A_a = \overline{aAa}$ is a closed ideal.*

Multiplicative functional III

- An element a of A is positive if a is hermitian and $\sigma(a) \subset \mathbb{R}_+$. A subalgebra B of A is said to be hereditary if for $a \in A_+$ and $b \in B_+$ the inequality $a \leq b$ implies $a \in B$.
- We say a state f on A is pure if it has the property that whenever g is a positive linear functional on A such that $g \leq f$, necessarily there is a number $t \in [0,1]$ such that $g = tf$.

Theorem 8

If the Jacobson radical $\mathfrak{R}$ of Banach algebra X is commutative (particularly, $\mathfrak{R} = 0$), X is commutative if and only if any left(right) closed ideal is a closed ideal.

Corollary 9 (Jacobson)

If $x^n = x$ for some integer $n > 1$ for any $x \in X$, then X is commutative. [Hint: Take quotient $X/\mathfrak{p}$ and find every prime ideal is maximal. Use 8.]

Multiplicative functional IV

Proof.

Only left to prove the necessity. By Zorn's lemma, any Banach algebra X that has at least two elements contains a maximal left ideal m. Sketch:

- any non-zero element $a \in X/\mathrm{m}$ has a left inverse;
- any non-zero element $a \in X/\mathrm{m}$ has an inverse;
- $X/\mathrm{m} \cong \mathbb{C}$ (Gelfand-Mazur);
- the following sequence is exact

$$0 \to \mathfrak{R}(= \ker \Gamma) \xrightarrow{i} X \xrightarrow{\Gamma} C(\mathfrak{M})$$

  where i is identity map and Γ is the Gelfand transformation.

Suppose there exists a non-zero left inevitable $b \in B = X/\mathrm{m}$, $\mathrm{b} = \overline{Bb}$ is a left closed ideal of B. This leads to a contradiction since $e_B \notin \mathrm{b}$. □

Multiplicative functional IV

$$\begin{tikzcd}
\mathsf{a} = \bar{i}^{-1}(\mathsf{b}) \arrow[r, "\bar{i}"] \arrow[d] & \mathsf{b} \arrow[d] \\
X \arrow[r, "i"] & B
\end{tikzcd}$$

Figure: The ideal corespondence theorem.

Example 10

The quaternion $\mathbb{H}$ is a real Banach division algebra, which is not commutative.

Question 1

Does the theorem still hold true when the commutative assumption of $\mathfrak{R}$ is dropped out?

Hierarchy of Maximal ideals I

Let $\mathfrak{M}$ denote the set of maximal ideals of X, and equip $\mathfrak{M}$ with weak*-topology as follows

$$U_x(\mathsf{m}_0, 1/n) = \{\mathsf{m} \in \mathfrak{M} : |x(\mathsf{m}_0) - x(\mathsf{m})| < 1/n\}, n = 1, 2, 3, \cdots$$

The finite intersections of $U_{x_i}(\mathsf{m}_0, 1/n), i = 1, 2, \ldots, k$ constitude a neighborhood basis for $\mathsf{m}_0 \in \mathfrak{M}$.

Theorem 11

$\mathfrak{M}$ is a compact T_2 space.

Proof of T_2.

For any $\mathsf{m}, \mathsf{m}' \in \mathfrak{M}$, $\mathsf{m} \neq \mathsf{m}'$, there exists $x \in X$ such that $x(\mathsf{m}) \neq x(\mathsf{m}')$. $U_x(\mathsf{m}, \varepsilon) \cap U_x(\mathsf{m}', \varepsilon) = \emptyset$ for suitable ε. $\square$

Hierarchy of Maximal ideals II

Proof of compactness.

By Tychonoff theorem, the unit ball in X^* is weak*-compact. Define $A = \{x \in X : x(\mathsf{m}) = 0\}$ where $\mathsf{m} \in X^*$. Thus for any $a \notin A$, a and A spans X. For any $x, y \in A$ and $1/n$, $|x_n y_n(\mathsf{m}_n) - xy(\mathsf{m})| < 1/n$. By $|x + y| \leq |x| + |y|$, the absolute value of $xy(\mathsf{m})$ can be made arbitarily small, so $xy(\mathsf{m}) = 0$, i.e. $xy \in A$. $\square$

Example 12

If X is separable, then $\mathfrak{M}$ has countable topological basis. Hence if X is separable, then $\mathfrak{M}$ is metrizable.

Example 13

If X is finitely generated, then $\mathfrak{M}$ is homeomorphic to a compact subspace of $\mathbb{C}^n$. ($\mathfrak{M}$ is isomorphic to the unit ball of $\mathbb{C}^n$.)

Hierarchy of Maximal ideals III

Let $\mathfrak{N}$ denote the set of prime ideals of X(unital!), a topological basis of $\mathfrak{N}$ is given as follows

$$\tilde{U}_x = \{\mathsf{p} \in \mathfrak{N} : x \notin \mathsf{p}\}$$

We know that

Example 14

1. $\mathfrak{N}$ is T_0. Let $V(\mathsf{p})$ denote the smallest closed set containing p. Evidently,

$$y \in \overline{\{x\}} \Leftrightarrow \overline{\{y\}} \subseteq \overline{\{x\}} \Leftrightarrow V(\mathsf{p}_y) \subseteq V(\mathsf{p}_x) \Leftrightarrow \mathsf{p}_x \subseteq \mathsf{p}_y.$$

   Since $\mathsf{p} \subset \mathsf{p}_0$ shows $V(\mathsf{p}) \subset V(\mathsf{p}_0)$, $V(\mathsf{p}_0)^c$ is an open set containing p and $\mathsf{p} \notin V(\mathsf{p}_0)^c$. If not, $V(\mathsf{p})^c$ is the open subset containing p_0 and silmiar. Moreover,
2. $\mathfrak{N}$ is compact. Since 1 is finitely generately from the open covering.

Hierarchy of Maximal ideals IV

Theorem 15

The topology of $\mathfrak{M}$ inherited from $\mathfrak{N}$ as a subspace is homeomorphic to weak-topology.*

Theorem 16

The topology of $\mathfrak{M}$ inherited from $\mathfrak{N}$ as a subspace is T_1.

Proof.

For any $\mathrm{m}, \mathrm{m}' \in \mathfrak{M}$, $\tilde{U}_x$ where $x \in \mathrm{m}$ and $x \notin \mathrm{m}'$ is such an open set. Same for the other. □

Theorem 17

The topology of $\mathfrak{M}$ inherited from $\mathfrak{N}$ as a subspace is T_2.

Work in ZFC

Koszmider et al. have proved in Fund. Math. 254 (2021) p.15–47 that

Theorem 18

It is consistent with ZFC that every C-algebra of $\aleph_1$ generators contains a nonseparable commutative subalgebra.*

Theorem 19

It is independent from ZFC whether there is an C-algebra of $\aleph_1$ generators with no nonseparable commutative subalgebra.*

These are closely related to the diamond $\diamondsuit$ principle and Naimark problem.

Mazur's and Naimark's problems I

Question 2 (Mazur, 1932)

Does any infinite-dimensional Banach space have an infinite-dimensional separable Hausdorff quotient?

Example 20

Infinite-dimensional reflexive Banach spaces.

The following argument is due to Rosenthal and Lacey.

(1) K is scattered. Then K contains a convergent sequence (x_n) of distinct points. The linear map $T : C(K) \to c$, $f \mapsto (f(x_n))$ is a continuous surjection. Hence the quotient $C(K)/T^{-1}(0)$ is isomorphic to c.

(2) K is not scattered. Then K is continuously mapped onto $[0, 1]$. The space l_2 is isomorphic to a closed subspace of $L[0, 1]$ and $l_1[0, 1]$ is isomorphic to a closed subspace $L_1(K, \mathfrak{B}_K, \mu)$, where μ is some non-negative finite regular Borel measure on K. The latter space is isomorphic to a closed subspace of the norm dual Y of $C(K)$. Therefore the reflexive space l_2 is a subspace of Y that is weakly*-closed, and then a quotient of $C(K)$ is isomorphic to l_2.

Question 3 (Naimark, 1951)

Is every C-algebra that has only one irreducible *-representation up to unitary equivalence isomorphic to the *-algebra of compact operators on some (not necessarily separable) Hilbert space.*

Two simple but interesting open problems

We say that $D = \{V_\beta\}_{\beta \in B}$ is a refinement of $C = \{U_\alpha\}_{\alpha \in A}$ if for all $\beta \in B$ there exists $\alpha \in A$ such that $V_\beta \subseteq U_\alpha$.

Question 4 (Krupski, 2015, IJM)

Suppose that K is an uncountable compact metrizable space (there exists open cover that doesn't have an open refinement). Is it true that $C_p(K)$ and $C_w(K)$ are not homeomorphic?

We say that a bounded linear operator $T : C(K) \to C(K)$ is a weak multiplication, if $T = gI + S$, where g is a continuous function on K and S is weakly compact. T is called a weak multiplier, if $T^* = gI + S$ for some bounded Borel function g and weakly compact S.

Question 5 (Glodkowski, 2023, JFA)

Suppose that K is a compact Hausdorff space such that every operator is a weak multiplier and $C(L) \cong C(K)$ for some compact Hausdorff space L. Is it true that K and L are homeomorphic modulo finitely many points?

Thank you!

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