# Introduction to Stability Theory

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# First-order logic

The first-order logic, also known as predicate calculus, mainly focuses on the separation of semantics and syntax. In a countable language of symbols $\mathcal{L}$, for example, the language for semi-groups $\mathcal{L} = \{\cdot, e\}$, there is an $\mathcal{L}$-model $\mathfrak{M} = \langle M, \sigma \rangle$ or an $\mathcal{L}$-structure representative for a concrete semi-group where $\sigma : f \mapsto A \subseteq M^{n_f}$ maps each symbol from the language $\mathcal{L}$, such as constants, relations, functions and $n_f$ is the arity for the symbol $f$. By $\mathrm{Th}(\mathfrak{M})$ we mean the full theory of $\mathcal{L}$-sentences satisfied by $\mathfrak{M}$, namely $\mathfrak{M} \models \phi$ iff $\phi \in \mathrm{Th}(\mathfrak{M})$. Note that an $\mathcal{L}$-sentence is an $\mathcal{L}$-formula without free variables. Also, $T = \mathrm{Th}(\mathfrak{M})$ is a complete theory in the sense that either $T \models \phi$ or $T \models \neg\phi$ for any $\mathcal{L}$-sentence $\phi$. A definable subset $X \subseteq M^n$ for some $n$ is one that can be encoded by an $\mathcal{L}$-formula $\phi(\bar{x}, \bar{y}), \bar{x} \in M^n, \bar{y} \in M^m$ with a parameter $\bar{b} \in M^m$ for some $m$ such that $X = \{\bar{a} \in M^n : \mathfrak{M} \models \phi(\bar{a}, \bar{b})\}$. Specifically, $X$ is definable over a subset $A \subseteq M$ if $\bar{b} \in A^m$. Let $\mathrm{dcl}(A)$ be those elements $a \in M$ such that $\{a\}$ is $A$-definable.

# Shelah's universe Meq I

In first-order logic, the set $M$ is the domain or universe for objects we take into consideration. It's sometimes more convenient to consider a partitioned universe $M = \sqcup_{i \in I} M_i$. By this convention, we collect all $\emptyset$-definable equivalence relations $\sim$ on $M^n$ for some $n$ and construct a universe $M^{\mathrm{eq}}$ to be the disjoint union of equivalence classes $M^n / \sim$. Because $=$ is a $\emptyset$-definable equivalence relation on $M$, we can identify $M$ with some partition in $M^{\mathrm{eq}}$. An element $a \in M$ is algebraic over $A$ if there exist an $\mathcal{L}$-formula $\phi(x, \bar{y})$ and $\bar{b} \in A^m$ such that $\mathfrak{M} \models \phi(a, \bar{b})$ and the set $\{x \in M : \mathfrak{M} \models \phi(x, \bar{b})\}$ is finite. Similarly, let $\mathrm{acl}(A)$ be those elements algebraic over $A$. Recall a theory $T$ is a set of $\mathcal{L}$-sentences. $T$ eliminates quantifier if for every formula $\phi$ there is a quantifier-free formula $\psi$ such that $T \models \phi \leftrightarrow \psi$ where free variables are taken universal closures $\forall \bar{x} \phi(\bar{x})$.

# Shelah's universe Meq II

## Definition (Poi00)

A theory $T$ eliminates imaginaries if for every $a \in M^{\mathrm{eq}}$ there exists $\bar{b} \in M^m$ for some $m$ such that $\mathrm{dcl}^{\mathrm{eq}}(a) = \mathrm{dcl}^{\mathrm{eq}}(\bar{b})$ and weakly eliminates imaginaries if $a \in \mathsf{dcl}^{\mathrm{eq}}(\bar{b}) \wedge \bar{b} \in \mathsf{acl}^{\mathrm{eq}}(a)$.

## Theorem (Shelah90)

$T^{\mathrm{eq}} = \mathrm{Th}(\mathfrak{M}^{\mathrm{eq}})$ *eliminates imaginaries.*

# Shelah's universe Meq III

## Proof.

Fix a regular cardinal $\kappa > |\mathcal{L}|$ and a $\kappa$-big model $\mathfrak{C} \models T$. [Hod93] Let $E$ be a $\emptyset$-definable equivalence relation in $\mathfrak{C}^{\mathrm{eq}}$. Suppose the domain of $E$ consists of $n$-tuples of sort $S_{E'}$ where $E'$ defines an equivalence relation on $\mathfrak{C}^k$. By Lemma 1.4(iii) there is an $\mathcal{L}$-formula $\psi(\bar{x}_1, \ldots, \bar{x}_n, \bar{y}_1, \ldots, \bar{y}_n)$ such that for $k$-tuples $\bar{a}_1, \ldots, \bar{a}_n, \bar{b}_1, \ldots, \bar{b}_n$ from $\mathfrak{C}$, $\mathfrak{C} \models \psi(\bar{a}_1, \ldots, \bar{a}_n, \bar{b}_1, \ldots, \bar{b}_n)$ iff $\mathfrak{C} \models E(\bar{a}_1/E', \ldots, \bar{a}_n/E', \bar{b}_1/E', \ldots, \bar{b}_n/E')$. But then the formula $\psi$ defines an equivalence relation on $\mathfrak{C}^{nk}$. Let $X$ be an $E$-equivalence class. Let $(a_1, \ldots, a_n) \in X$, and let $\bar{a}_i$ be some $k$-tuple from $\mathfrak{C}$ such that $\bar{a}_i/E' = a_i$. Then the element $(a_1, a_2, \ldots, a_n)/\psi$ of sort $S_\psi$ is a code for $X$. $\square$

# Complete and partial Types

Let $\mathcal{L}_A$ be the language obtained by adding to $\mathcal{L}$ constant symbols for each $a \in A \subseteq M$. An $n$-type $p$ over $A$ is a set of $\mathcal{L}_A$-formulas in $n$ free variables such that $p \cup \mathrm{Th}_A(\mathfrak{M})$ is satisfiable. A type $p$ is complete if either $\phi \in p$ or $\neg\phi \in p$ for all $\mathcal{L}_A$-formula $\phi$. Denote the set of all complete $n$-types as $S_n^{\mathfrak{M}}(A)$ and omit $\mathfrak{M}$ if possible. For $\bar{a} \in M^n$, let $\mathrm{tp}(\bar{a}/A) = \{\phi \in \mathcal{L}_A : \mathfrak{M} \models \phi(\bar{a})\}$ and write $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{a}/\emptyset)$ for simplicity. Consider another $\mathcal{L}$-structure $\mathfrak{N} = \langle N, \tau \rangle$. An injective map $j : M \to N$ is an elementary embedding if $\tau = j \circ \sigma$ and $\mathfrak{M} \models \phi(a)$ iff $\mathfrak{N} \models \phi(j(a))$ for all $\mathcal{L}$-formulas $\phi(x)$ and $a \in M$ and we write $\mathfrak{M} \prec \mathfrak{N}$ for this. Similarly, an injective map $f$ from $B \subseteq M$ to $N$ is called partial elementary if $\tau = f \circ \sigma$ and $\mathfrak{M} \models \phi(a)$ iff $\mathfrak{N} \models \phi(f(a))$ for all $a \in B$ and $\phi$. Define a basic open subset on $S_n(A)$ by $[\phi] = \{p \in S_n(A) : \phi \in p\}$. This topology is compact, Hausdorff and totally disconnected. A type $p$ is isolated if $\{p\} = [\phi]$. We say that $\bar{a} \in M^n$ realizes an $n$-type $p$ if $\mathfrak{M} \models \phi(\bar{a})$ for all $\phi \in p$. If $p$ is not realized by any $\bar{a} \in M^n$, we say $\mathfrak{M}$ omits it.

# Saturation and Homogeneity

## Definition (Mar13 & Poi00)

1. $\mathfrak{M}$ is called $\kappa$-homogeneous if whenever $A \subseteq M$, $|A| < \kappa$, $f : A \to M$ is a partial elementary map, and $a \in M$, there is a partial elementary map $f^* : A \cup \{a\} \to M$ such that $f^* \supseteq f$.
2. $\mathfrak{M}$ is called strongly $\kappa$-homogeneous if every partial elementary map extends to an automorphism.
3. $\mathfrak{M}$ is called $\kappa$-saturated if for every $A \subseteq M$ with $|A| < \kappa$, every complete $n$-type $p \in S_n(A)$ is realized in $M$, for every $n < \omega$.
4. $\mathfrak{M}$ is called $\kappa$-universal if for every $\mathfrak{N} \models T$ with $|N| < \kappa$, there is an elementary embedding $j : N \to M$. We say that $\mathfrak{M}$ is saturated (homogenous) if it is $|M|$-saturated (homogenous) and that $\mathfrak{M}$ is universal if it is $|M|^+$-universal.

## Theorem (Hod90)

*There exists an elementary extension $\mathfrak{N}$ such that $|N| \leq |M|^{<\kappa}$ that is both $\kappa$-saturated and strongly $\kappa$-homogeneous.*

# Forking I

Let $\delta(\bar{x}, \bar{y})$ be an $\mathcal{L}$-formula. A complete $\delta$-type over $A$ is a maximal consistent set $p(\bar{x}) \subseteq \{\delta(\bar{x}, \bar{a})^t : \bar{a} \in A^m, t \in \{0,1\}\}$. The set of all complete $\delta$-types over $A$ is denoted by $S_\delta(A)$. A formula $\delta$ is stable if there exists some $\lambda$ such that for all $A \subseteq M, |A| < \lambda$ and $n < \omega$, $|S_\delta(A)| < \lambda$. It has the order property if there are tuples $(\bar{a}_i)_{i<\omega}$ and $(\bar{b}_i)_{i<\omega}$ such that $\mathfrak{M} \models \delta(\bar{a}_i, \bar{b}_j) \Leftrightarrow i \leq j$. By a $\delta$-definition of $p(\bar{x}) \in S_\delta(A)$ we mean an $\mathcal{L}_M$-formula $d_p\phi(\bar{y})$ such that for all $\bar{b} \in M$, $\phi(\bar{x}, \bar{b}) \in p(\bar{x})$ iff $\mathfrak{M} \models d_p\phi(\bar{b})$. A sequence of distinct elements $(\bar{a}_i)_{i\in I}$ is order-indiscernibles over $A$ if for all $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_m$ from an ordered set $I$, $\mathfrak{M} \models \phi(a_{i_1}, \ldots, a_{i_m}) \leftrightarrow \phi(a_{j_1}, \ldots, a_{j_m})$. A formula $\delta(\bar{x}, \bar{a})$ divides over $A$ if there is an $A$-indiscernible sequence $(\bar{a}_i)_{i<\omega}$ with $\bar{a}_0 = \bar{a}$ such that $\{\delta(\bar{x}, \bar{a}_i)\}_{i<\omega}$ is inconsistent. The formula $\delta$ forks over $A$ if it is a disjunction of dividing formulas. For types, this holds for each formula. The restriction of a complete $n$-type $p(\bar{x})$ over $B$ to $A$ is denoted as $p \restriction A \in S_n(A)$. Similarly, $p \restriction \delta \in S_\delta(A)$ for some formula $\delta(\bar{x}, \bar{y})$ consists of all $\delta$-formulas in $p(\bar{x})$. A complete type

# Forking II

$p(\bar{x}) \in S_n(A)$ is stationary, if $p$ has a unique non-forking extension over any set $B \supseteq A$, so $\mathrm{stp}(\bar{a}/A) = \mathrm{tp}(\bar{a}/\mathrm{acl}^{\mathrm{eq}}(A))$ is stationary. If $p$ is a stationary type, its canonical base $\mathrm{Cb}(p)$ is the definable closure in $\mathfrak{C}^{\mathrm{eq}}$ of all definitions $d_p\phi(\bar{y})$ as $\phi$ ranges over $p$.

## Theorem (Existence)

*A non-forking extension exists, and is unique if $A = \mathrm{acl}(A)$.*

Proof. Let $p_1(\bar{x}) \in S_n(\mathrm{acl}(A))$ be any extension of $p(\bar{x})$. From the definition of non-forking it suffices to prove that $p_1$ has a unique non-forking extension in $S_n(M)$. For each stable formula $\delta(\bar{x}, \bar{y})$, let $q_\delta(\bar{x}) \in S_\delta(M)$ be the unique non-forking extension of $p_1 \restriction \delta$ by 2.9. This is assured since for each finite set $\Delta$ of such formulae $\delta$, there is by 2.18 a non-forking extension $q_\Delta \in S_\Delta(M)$ of $p_1 \restriction \Delta$. In particular $q \restriction \Delta$ is a non-forking extension of $p_1 \restriction \delta$ for each $\delta \in \Delta$. By uniqueness 2.9, $q_\Delta \restriction \delta = q_\delta$ for each $\delta \in \Delta$. Thus, $\cup\{q_\delta(\bar{x})\}$ is consistent and determines a complete type in $S_n(M)$, which is the unique non-forking extension of $p_1(\bar{x})$. $\square$

# Forking III

## Theorem (Definability)

*A formula $\delta$ is stable iff any complete $\delta$-type $p(\bar{x}) \in S_\delta(A)$ has a $\delta$-definition.*

## Lemma (Pil93, Lemma 2.2)

*If $\delta(\bar{x}, \bar{y})$ is stable, then $p(x)$ has a $\delta$-definition $d_p\psi(\bar{y})$ which is a positive Boolean combination of formulae $\delta(\bar{c}, \bar{y}), \bar{c} \in M^n$.*

## Lemma (Mar13, Theorem 5.2.3)

*If $\delta(\bar{x}; \bar{y})$ has the order property, then for every linear order $I$, there are order-indiscernibles $(\bar{a}_i, \bar{b}_i)_{i \in I}$ preserving order property.*

## Lemma (Mar13, Exercise 5.5.6)

*A formula is unstable iff it has the order property.*

# Forking IV

## Lemma (Finite alternation)

*If $\delta(\bar{x}; \bar{y})$ is stable, then there exists $N < \omega$ such that for every indiscernible sequence $(\bar{b}_i)_{i<\omega}$ and every tuple $\bar{c}$, the truth values $\models \delta(\bar{c}; \bar{b}_i)$ alternate at most $N$ times.*

Proof. If no such $N$ existed, by compactness theorem we have an indiscernible sequence on which the truth values of $\delta(\bar{c}; \bar{b}_i)$ alternate infinitely often. It contradicts the stability since from the thinning lemma this yields an order-indiscernible sequence witnessing the order property. $\square$

Proof of Definability. First assume that $\delta$ is stable. It suffices to show that every $\delta$-type over $A$ is definable. Let $p(\bar{x}) \in S_\delta(A)$. Define

$$X_p = \{\bar{a} \in A^m : \delta(\bar{x}; \bar{a}) \in p\}$$

which is definable by the finite Boolean lemma. Since $\delta$ is stable, truth values of $\delta(\bar{c}; \bar{b}_i)$ along an indiscernible sequence $(\bar{b}_i)_{i<\omega}$ are eventually constant uniformly in $\bar{c}$.

# Forking V

Conversely, suppose that every complete $\delta$-type is definable. We show that $\delta$ is stable. If $\delta$ were unstable, then $\delta$ would have the order property. By the thinning lemma, we may assume that there is an order-indiscernible sequence $A = (\bar{b}_i)_{i \in \mathbb{Q}}$ and tuples $\bar{a}_i, i \in \mathbb{Q}$ such that $\models \delta(\bar{a}_i, \bar{b}_j) \Leftrightarrow i > j$. Choose an irrational cut $\alpha$ of $\mathbb{Q}$, and consider the partial $\delta$-type

$$q(\bar{x}) = \{\delta(\bar{x}, \bar{b}_i) : i < \alpha\} \cup \{\neg\delta(\bar{x}, \bar{b}_i) : i > \alpha\}.$$

Every finite subset of $q$ is satisfiable, so $q$ extends to some complete $\delta$-type $p(\bar{x}) \in S_\delta(A)$. By assumption, $p$ has a $\delta$-definition $d_p\delta(\bar{y})$. This formula uses only finitely many parameters from $A$, say $\bar{b}_{i_1}, \ldots, \bar{b}_{i_m}$. Choose $r < \alpha < s$ such that $r$ and $s$ lie in the same interval determined by $i_1, \ldots, i_m$. such that $\models d_p\delta(\bar{b}_r)$ iff $\models d_p\delta(\bar{b}_s)$. But $\delta(\bar{x}, \bar{b}_r) \in p \wedge \neg\delta(\bar{x}, \bar{b}_s) \in p$. Contradiction. $\square$

# Forking VI

### Theorem (She90, Theorem III.3.12)

*If $T$ is complete and $\mathcal{L}$ is countable, then one of the three holds*

1. *there are no cardinals $\kappa$ such that $T$ is $\kappa$-stable*
2. *$T$ is superstable in the sense that $\kappa$-stable for all $\kappa \geq 2^{\aleph_0}$*
3. *$T$ is not superstable but $\kappa$-stable for all $\kappa^{\aleph_0} = \kappa$*

We say that $A \subseteq D$ is independent if $a \notin \mathrm{acl}(A \setminus \{a\})$ for all $a \in A$. Also, $A$ is independent over $C \subseteq D$ if $a \notin \mathrm{acl}(C \cup (A \setminus \{a\}))$ for all $a \in A$. A sequence $(a_\alpha)_{\alpha<\lambda}$ for a complete n-type $p(\bar{x})$ over $A$ is a Morley sequence of lenght $\lambda$ if for each $\alpha < \lambda$, $a_\alpha$ realizes $p \restriction A \cup \{a_\beta : \beta < \alpha\}$.

### Theorem (Pil93, Lemma 2.28)

*A Morley sequence $(a_i)_{i<\omega}$ over $A$ is $A$-independent and $A$-indiscernibles.*

# Generic types I

Let $D \subseteq M^n$ be an infinite definable set. We say that $D$ is minimal if for any definable $Y \subseteq D$ either $Y$ is finite or $D \setminus Y$ is finite. If $\phi(\bar{x}, \bar{a})$ is the formula that defines $D$, then we also say that $\phi(\bar{x}, \bar{a})$ is minimal. We say that $D$ and $\phi$ are strongly minimal if $\phi$ is minimal in any elementary extension of $\mathfrak{M}$. We say that a theory $T$ is strongly minimal if the formula $v = v$ is strongly minimal. A generic type $p_D = \mathrm{tp}(a/A)$ is the type of a point of $a \in D$ which avoids every proper finite $A$-definable subset of $D$. A type $p$ is minimal if $p$ is non-algebraic and every forking extension of $p$ is algebraic. Use the symbol $a \downarrow_A B$ for $a \in M$ independent from $B$ over $A$ if $\mathrm{tp}(a/A \cup B)$ does not fork over $A$ and write $C \downarrow_A B$ by the elementary-wise meaning.

# Generic types II

## Example (Generic points in a parabola)

Consider the irreducible curve

$$V = \{(x, y) \in K^2 : y = x^2\}.$$

Its ideal is

$$I(V) = (y - x^2) \subseteq K[x, y].$$

The generic point of $V$ is represented by $(t, t^2)$, where $t$ is transcendental over $K$. It satisfies the equation $y = x^2$, but it satisfies no extra algebraic equation that is not identically true on $V$. Equivalently, if

$$g(x, y) \notin (y - x^2),$$

then $g(t, t^2) \neq 0$. Thus the generic point lies on the parabola, but avoids every proper algebraic subset of the parabola.

# Generic types III

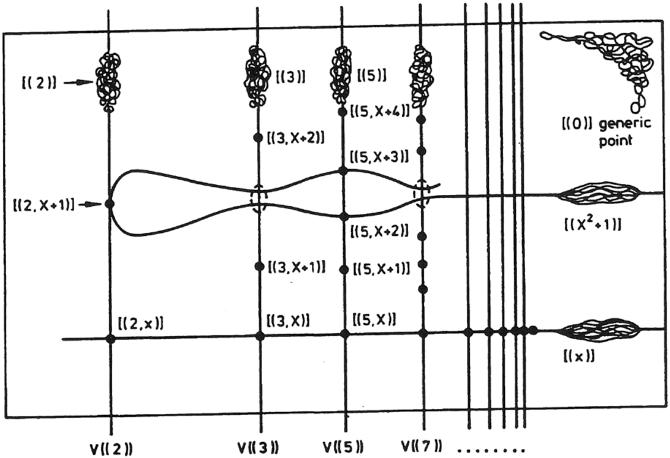


Figure 1: Generic points in $\operatorname{Spec}\mathbb{Z}[x]$. The Red Book of Mumford, pp.75.

# DLO I

### Example (Dense Linear Orders without Endpoints)

The theory of dense linear orders without endpoints is axiomatized in the language $\mathcal{L}_< = \{<\}$, which is complete, eliminates quantifiers, and is $\aleph_0$-categorical, but it is not stable. For example, the structure $(\mathbb{Q}, <)$ is the unique countable model of $\mathrm{DLO}$ up to isomorphism. Hence $\mathrm{DLO}$ is $\aleph_0$-categorical. Cleary, $\mathrm{DLO}$ has the order property, so it is unstable. In fact, $|S_1(\mathbb{Q})| = 2^{\aleph_0}$ exactly.

### Lemma (Back-and-forth)

*Let $\mathfrak{M} = \langle M, < \rangle$ and $\mathfrak{N} = \langle N, < \rangle$ be countable models of $\mathrm{DLO}$, and let $A \subseteq M$ be finite. If $f : A \to N$ is a partial order-embedding, then for every $a \in M$, there exists a partial order-embedding $f^* : A \cup \{a\} \to N$ such that $f^* \supseteq f$ and vice versa.*

# DLO II

## Lemma (Categoricity)

$\mathrm{DLO}$ *is* $\aleph_0$*-categorical in the sense that any two countable models of* $\mathrm{DLO}$ *are order-isomorphic.*

Proof. Enumerate $M = \{a_0, a_1, \ldots\}$ and $N = \{b_0, b_1, \ldots\}$. We construct an increasing sequence of partial order-isomorphisms $f_0 \subseteq f_1 \subseteq \cdots$. Start with $A_0 = B_0 = f_0 = \varnothing$. Suppose $f_s : A_s \to B_s$ has already been constructed.
(i) If $s = 2n$, we extend the domain so that $a_n \in A_{s+1}$. If $a_n \in A_s$, let $f_{s+1} = f_s$. Otherwise define

$$L = \{c \in A_s : c < a_n\}, \quad U = \{c \in A_s : a_n < c\}.$$

Then every element of $f_s(L)$ is below every element of $f_s(U)$. Since $A_s$ is finite, if both $L$ and $U$ are nonempty, choose

$$m = \max f_s(L), \quad n = \min f_s(U).$$

# DLO III

By density of $N$, choose $b \in N$ such that $m < b < r$. If $L = \varnothing$, choose $b \in N$ below all elements of $f_s(U)$. If $U = \varnothing$, choose $b \in N$ above all elements of $f_s(L)$. This is possible because $N$ has no endpoints. Define $f_{s+1} = f_s \cup \{(a_n, b)\}$. Then $f_{s+1}$ is again a partial order-isomorphism.
(ii) If $s = 2n + 1$, we extend the range so that $b_n \in B_{s+1}$. If $b_n \in B_s$, let $f_{s+1} = f_s$. Otherwise apply the same argument with $M$ and $N$ interchanged.
(iii) Now let $f = \bigcup_{s<\omega} f_s$. Every $a_n \in M$ is put into the domain at some even stage, and every $b_n \in N$ is put into the range at some odd stage. Hence $f$ is a bijection $M \to N$. Since each $f_s$ preserves the order, so does $f$. Therefore $f : M \to N$ is an isomorphism. □

## Corollary (Completeness)

$\mathrm{DLO}$ *is complete.*

Proof. Because there are no finite dense linear orders, Vaught's test implies that $\mathrm{DLO}$ is also complete. □

# DLO IV

## Theorem (Quantifier-elimination)

$\mathrm{DLO}$ *eliminates quantifiers.*

Proof. Choose a countable model $(\mathbb{Q}, <) \models \mathrm{DLO}$. First suppose that $\phi$ is a sentence. If $(\mathbb{Q}, <) \models \phi$, then $\mathrm{DLO} \models \phi$ since $\mathrm{DLO}$ is complete. Thus $\mathrm{DLO} \models \phi \leftrightarrow x = x$. If $(\mathbb{Q}, <) \models \neg\phi$, this is similar. Now let $\phi(x_1, \ldots, x_n)$ be a formula with $n$ free variables. For each function $\sigma : \{(i, j) : 1 \leq i < j \leq n\} \to \{0, 1, 2\}$, define the corresponding sign condition $\chi_\sigma(x_1, \ldots, x_n)$ by

$$\chi_\sigma(\bar{x}) = \bigwedge_{\sigma(i,j)=0} x_i = x_j \wedge \bigwedge_{\sigma(i,j)=1} x_i < x_j \wedge \bigwedge_{\sigma(i,j)=2} x_i > x_j.$$

Each $\chi_\sigma$ is quantifier-free, and it describes a possible finite order pattern among $x_1, \ldots, x_n$. Let $\Lambda_\phi$ be the set of all sign conditions $\sigma$ such that there exists $\bar{a} \in \mathbb{Q}^n$ with

$$(\mathbb{Q}, <) \models \chi_\sigma(\bar{a}) \wedge \phi(\bar{a}). \tag{*}$$

# DLO V

Since there are only finitely many sign conditions, $\Lambda_\phi$ is finite. If $\Lambda_\phi = \varnothing$, then no tuple in $\mathbb{Q}^n$ satisfies $\phi$. Hence $(\mathbb{Q}, <) \models \neg\phi(\bar{x})$ and we may take $\psi(\bar{x}) = x_1 \neq x_1$. Now suppose $\Lambda_\phi \neq \varnothing$. Define a quantifier-free formula

$$\psi(\bar{x}) = \bigvee_{\sigma \in \Lambda_\phi} \chi_\sigma(\bar{x}).$$

First $(\mathbb{Q}, <) \models \phi(\bar{x}) \to \psi(\bar{x})$ is clear. Suppose $(\mathbb{Q}, <) \models \psi(\bar{b})$. Then for some $\sigma \in \Lambda_\phi$ we have $(\mathbb{Q}, <) \models \chi_\sigma(\bar{b})$. By the definition of $\Lambda_\phi$, there exists $\bar{a} \in \mathbb{Q}^n$ such that (*) holds. Thus $\bar{a}$ and $\bar{b}$ have the same finite order pattern. By the back-and-forth lemma, there is an order-isomorphism $f \in \mathrm{Aut}(\mathbb{Q}, <)$ such that $f(\bar{a}) = \bar{b}$. By coincidence lemma, $(\mathbb{Q}, <) \models \phi(\bar{b})$. Thus $(\mathbb{Q}, <) \models \psi(\bar{x}) \to \phi(\bar{x})$. Together with the first one, we prove the quantifier-elimination. $\square$

# DLO VI

## Theorem (Types)

*Let $\mathfrak{M} \models \mathrm{DLO}$, and let $p \in S_1(A)$. A cut of $A \subseteq M$ is a pair $(L, U)$ such that $A = L \sqcup U$ and $a < b$ for all $a \in L$ and $b \in U$.. Define*

$$L_p = \{a \in A : a < x \in p\}, \quad U_p = \{a \in A : x < a \in p\}.$$

*If $p$ is realized by some $a \in A$, then $p$ is isolated by $x = a$. Otherwise, if $p$ is not realized in $A$, then $p$ is isolated exactly when*

1. *$L_p$ has a greatest element $a$ and $U_p$ has a least element $b$. Then $\{p\} = [a < x < b]$.*
2. *$U_p = \varnothing$ and $L_p$ has a greatest element $a$. Then $\{p\} = [a < x]$.*
3. *$L_p = \varnothing$ and $U_p$ has a least element $b$. Then $\{p\} = [x < b]$.*

Proof. For each $a \in A$, since $p$ is complete, exactly one of $x = a, x < a, a < x$ belongs to $p$. If $x = a \in p$, then $p = \mathrm{tp}(a/A)$, and $x = a$ isolates $p$. Suppose now that $p$ is not realized in $A$.

# DLO VII

Then for every $a \in A$, exactly one of $a < x$ and $x < a$ belongs to $p$. Hence $A = L_p \sqcup U_p$. If $a \in L_p$ and $b \in U_p$, then $p$ contains $a < x$ and $x < b$. Since $p \cup \mathrm{Th}_A(\mathfrak{M})$ is satisfiable, we must have $a < b$. Thus $L_p < U_p$, so $(L_p, U_p)$ is a cut of $A$. Conversely, let $(L, U)$ be a cut of $A$. Define a partial type over $A$

$$q = \{a < x : a \in L\} \cup \{x < b : b \in U\}.$$

Clearly, $q$ is satisfiable. By Zorn's lemma, $q$ extends to a complete type $q_1$. Since quantifier-free formulas are Boolean combinations of atomic formulas, the cut $(L, U)$ uniquely determines $q_1$. Suppose $p$ is isolated by $\phi$. If $U_p \neq \varnothing$ has no least element, choose $c \in U_p$ below the finitely many upper bounds used in the cut conditions. Then $\mathrm{tp}(c/A), c \in A$ realizes $[\phi]$, contradicting that $[\phi]$ isolates $p$. The opposite situation is similar. $\square$

## Corollary (Unstability)

*If* $|A| = \aleph_0$, $|S_1(A)| = 2^{\aleph_0}$, *so* $\mathrm{DLO}$ *is unstable.*

# ACF I

Let $T_\forall$ be the universal $\mathcal{L}$-theory consisting of all sentences $\forall \bar{x}\phi$ with $\phi$ quantifier-free in $T$. A theory $T$ has algebraically prime models if for any $\mathfrak{A} \models T_\forall$, there exist $\mathfrak{M} \models T$ and an embedding $i : \mathfrak{A} \to \mathfrak{M}$ such that for every $\mathfrak{N} \models T$ and every embedding $j : \mathfrak{A} \to \mathfrak{N}$, there exists an embedding $h : \mathfrak{M} \to \mathfrak{N}$ making the following diagram commute

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "i"] \arrow[dr, "j"'] & \mathfrak{M} \arrow[d, "h"] \\ & \mathfrak{N} \end{tikzcd}$$

The models $\mathfrak{M}$ are called algebraically prime over $\emptyset$. A class of $\mathcal{L}$-structures is axiomizable if they share a common $\mathcal{L}$-theory $T$ called axioms. An $\mathcal{L}$-theory $T$ is model-complete if every substructure of $\mathfrak{M} \models T$ can be elementarily embedded into $\mathfrak{M}$.

# ACF II

## Example (Algebraically Closed Fields)

Let $\mathcal{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$. Let $\mathrm{ACF}$ be the axioms for algebraically closed fields. If $p$ is a prime, $\mathrm{ACF}_p$ is obtained from $\mathrm{ACF}$ by adding the $p \cdot 1 = 0$ and so is $\mathrm{ACF}_0$ by adding all $\neg p \cdot 1 = 0$ for $p$ prime. Equivalently, $\mathrm{ACF}_p$ says that the field axioms hold, the characteristic is $p$, and every nonconstant polynomial has a root

$$\forall a_0, \ldots, a_{n-1} \exists x \big( x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \big).$$

The theory $\mathrm{ACF}_p$ is complete, eliminates quantifiers and imaginaries, is uncountably categorical and is stable. Not surprisingly, the theory $\mathrm{ACF}$ satisfies all these properties except for completeness since $\mathrm{ACF} \not\models p \cdot 1 = 0$ and $\mathrm{ACF} \not\models \neg p \cdot 1 = 0$.

## Lemma

$\mathrm{ACF}_\forall$ *is the theory of integral domains.*

# ACF III

Proof. Every algebraically closed field is an integral domain. The axioms saying that we have a commutative ring with 1, that $0 \neq 1$, and that there are no zero divisors,

$$\forall x \forall y (xy = 0 \to x = 0 \vee y = 0),$$

are universal sentences. Hence every model of $\mathrm{ACF}_\forall$ is an integral domain. Conversely, let $R$ be an integral domain. Then $R$ embeds into a fraction field $S^{-1}R, S = R \setminus \{0\}$ and further into its algebraic closure $R^{\mathrm{alg}}$,

$$R \to S^{-1}R \to R^{\mathrm{alg}}.$$

Thus $R$ is a substructure of a model of $\mathrm{ACF}$. Universal sentences are preserved under substructures. Therefore every universal sentence true in all algebraically closed fields is true in $R$. Hence $R \models \mathrm{ACF}_\forall$. So $\mathrm{ACF}_\forall$ is exactly the theory of integral domains. $\square$

# ACF IV

## Lemma (Algebraically prime models)

*Let $R \models \mathrm{ACF}_\forall$ be an integral domain. Then $R^{\mathrm{alg}}$ is algebraically prime over $R$.*

Proof. Let $K \models \mathrm{ACF}$ be any algebraically closed field containing $R$. Since $K$ is a field, the embedding $R \to K$ extends uniquely to an embedding $S^{-1}R \to K$. Since $K$ is algebraically closed, $R^{\mathrm{alg}}$ embeds into every model of $\mathrm{ACF}$ containing $R$, fixing $R$ pointwise. Therefore $R^{\mathrm{alg}}$ is algebraically prime over $R$. $\square$

## Lemma (Model-completeness)

$\mathrm{ACF}$ *is model-complete.*

Proof. Let $K \subseteq L$ be models of $\mathrm{ACF}$. We show that $K \prec L$. By Robinson's test [Mar13, Exercise 3.4.12], it is enough to show that every existential formula over $K$ realized in $L$ is already realized in $K$. Such a formula is equivalent to one of the form

$$\exists x \big( f_1(x,a) = \cdots = f_r(x,a) = 0 \wedge g_1(x,a) \neq 0, \ldots, g_s(x,a) \neq 0 \big),$$

# ACF V

where $a \in K$ and the $f_i, g_j$ are polynomials over $K$. Suppose this formula has a solution in $L$. Then the corresponding constructible set over $K$ has a point in an algebraically closed extension of $K$. By Hilbert's Nullstellensatz, a constructible set over an algebraically closed field $K$ has a point in some algebraically closed extension if and only if it already has a point in $K$. Therefore the formula has a solution in $K$. Thus $K$ is existentially closed in $L$. Hence $K \prec L$. So $\mathrm{ACF}$ is model-complete. $\square$

**Lemma (Mar13, Corollary 3.1.12)**

*Let $T$ be model-complete. Suppose that for every substructure $A$ of a model of $T$, there is a model $M_A \models T$ algebraically prime over $A$. Then $T$ eliminates quantifiers.*

**Theorem (Quantifier elimination)**

$\mathrm{ACF}$ *eliminates quantifiers.*

# ACF VI

## Theorem (Categoricity)

$\mathrm{ACF}_p$ *is* $\kappa$*-categorical for uncountable* $\kappa$ *but not* $\aleph_0$*-categorical.*

Proof. Let $K \models \mathrm{ACF}_p$, and let $k$ be the prime field, that is, $k = \mathbb{Q}$ for $p = 0$, or $k = \mathbb{F}_p$ for prime $p$. If $B$ is a transcendence basis of $K$ over $k$, then $K \cong \overline{k(B)}$. Thus algebraically closed fields of fixed characteristic are classified up to isomorphism by their transcendence degree over the prime field. If $|K| = \kappa > \aleph_0$, then $\mathrm{trdeg}(K/k) = \kappa$. Therefore all models of $\mathrm{ACF}_p$ of cardinality $\kappa$ are isomorphic. Hence $\mathrm{ACF}_p$ is $\kappa$-categorical for every uncountable cardinal $\kappa$. For each $n \in \mathbb{N} \cup \{\aleph_0\}$, the algebraic closure $\overline{k(t_1, \dots, t_n)}$ is a countable model of $\mathrm{ACF}_p$, where $t_1, \dots, t_n$ are algebraically independent over $k$. Two such fields are isomorphic if and only if they have the same transcendence degree over $k$. Hence $\mathrm{ACF}_p$ has many countable models and is not $\aleph_0$-categorical. $\square$

## Theorem (Elimination of imaginaries)

$\mathrm{ACF}_p$ *eliminates imaginaries.*

# ACF VII

Proof. By quantifier elimination, every definable subset of $K^n$ is constructible, that is, a Boolean combination of Zariski closed sets. Hence it is enough to show that constructible sets have canonical codes by real tuples. First consider a Zariski closed set $V \subseteq K^n$. It is determined by its ideal $I(V) \subseteq K[X_1, \ldots, X_n]$. Since $K[X_1, \ldots, X_n]$ is Noetherian, $I(V)$ is finitely generated. For a sufficiently large degree $d$, the finite-dimensional vector space

$$I(V)_{\leq d} = I(V) \cap K[X_1, \ldots, X_n]_{\leq d}$$

determines $I(V)$. This subspace is coded by its Plücker coordinates in a Grassmannian. Thus $V$ has a real canonical code. A constructible set is a finite Boolean combination of Zariski closed sets, so it is coded by the finite tuple of codes of the closed sets appearing in such a Boolean description. Finite unordered tuples are coded in algebraically closed fields by symmetric functions. Therefore every constructible set has a real code. Now let $E(x, y)$

# ACF VIII

be a definable equivalence relation and let $a \in K^m$. The imaginary element represented by $a$ is the class $[a]_E$. The class $X_a = \{x \in K^m : E(x, a)\}$ is definable, hence constructible, so it has a real code $c_a$. An automorphism fixes $c_a$ if and only if it fixes the set $X_a$, and this is equivalent to fixing the imaginary class $[a]_E$. Hence

$$[a]_E \in \mathrm{dcl}^{\mathrm{eq}}(c_a) \wedge c_a \in \mathrm{dcl}^{\mathrm{eq}}([a]_E).$$

Thus every imaginary is interdefinable with a real tuple. Therefore $\mathrm{ACF}_p$ eliminates imaginaries. $\square$

## Theorem (Types over subfields)

*Let $K \models \mathrm{ACF}_p$, and let $k \subseteq K$ be a subfield. Then $q \in S_n^K(k)$ maps to $I_q = \{f \in k[X_1, \ldots, X_n] : f(v) = 0 \in q\}$ bijectively.*

Proof. Let $q \in S_n^K(k)$. We first show that $I_q$ is a prime ideal. If $f, g \in I_q$, then $f(v) = 0$ and $g(v) = 0$ belong to $q$, so $(f + g)(v) = 0$ belongs to $q$. If $f \in I_q$ and $h \in k[X_1, \ldots, X_n]$, then

# ACF IX

$(hf)(v) = 0$ belongs to $q$. Thus $I_q$ is an ideal. If $fg \in I_q$, then $f(v)g(v) = 0 \in q$. Since fields are integral domains,

$$\mathrm{ACF}_p \models \forall v \left(f(v)g(v) = 0 \to (f(v) = 0 \vee g(v) = 0)\right).$$

Because $q$ is complete, either $f(v) = 0 \in q$ or $g(v) = 0 \in q$. Hence $I_q$ is prime. Conversely, let $P \in \operatorname{Spec} k[X_1, \ldots, X_n]$. Let $F = \left(k[X_1, \ldots, X_n]/P\right)^{\mathrm{alg}}$, and let $x_i = X_i/P \in F$. Then $F \models \mathrm{ACF}_p$, and for every $f \in k[X_1, \ldots, X_n]$, $f(x_1, \ldots, x_n) = 0 \Leftrightarrow f \in P$. Thus the type $q = \mathrm{tp}^F(x_1, \ldots, x_n/k)$ has $I_q = P$. This proves surjectivity. For injectivity, suppose $q_1, q_2 \in S_n^K(k)$ and $I_{q_1} = I_{q_2}$. By quantifier elimination, every formula over $k$ is equivalent to a Boolean combination of polynomial equations $f(v) = 0, f \in k[X_1, \ldots, X_n]$. Since $q_1$ and $q_2$ contain the same polynomial equations, they contain the same formulas. Hence $q_1 = q_2$. Thus, $S_n^K(k) \cong \operatorname{Spec} k[X_1, \ldots, X_n]$. $\square$

# ACF X

## Corollary (Stability)

$\mathrm{ACF}_p$ *is stable.*

Proof. Let $A \subseteq K \models \mathrm{ACF}_p$, and let $k$ be the subfield generated by $A$. Then $|k| \leq |A| + \aleph_0$. By quantifier elimination, types over $A$ are the same as types over $k$. By the previous theorem, complete $n$-types over $k$ correspond to prime ideals of $k[X_1, \ldots, X_n]$. The ring $k[X_1, \ldots, X_n]$ is Noetherian, so every ideal is finitely generated. Since $|k[X_1, \ldots, X_n]| = |k| + \aleph_0$, there are at most $|k| + \aleph_0$ many prime ideals. Hence, for every infinite cardinal $\kappa$, if $|A| \leq \kappa$, then $|S_n(A)| \leq \kappa$. Thus $\mathrm{ACF}_p$ is stable. $\square$

## Theorem (Strongly minimal)

*Let $K \models \mathrm{ACF}_p$, and let $A \subseteq K$. Every $A$-definable subset of $K$ is finite or cofinite.*

Proof. By quantifier elimination, every $A$-definable subset of $K$ is defined by a finite Boolean combination of polynomial equations

# ACF XI

$f(x) = 0, f \in A[x]$. A nonzero polynomial in one variable has only finitely many roots in a field. Therefore every Boolean combination of such equations is finite or cofinite. Hence every definable subset of $K$ is finite or cofinite. $\square$

# RCF I

### Example (Real closed field)

Let $\mathcal{L}_{\mathrm{or}} = \{<, +, -, \cdot, 0, 1\}$ obtained from from $\mathcal{L}_{<} \sqcup \mathcal{L}_{\mathrm{ring}}$. A real closed field is an ordered field such that

1. $-1$ is not a sum of squares
2. either $a$ or $-a$ is a square
3. every polynomial of odd degree has a root

Let $\mathrm{RCF}$ be the theory of real closed fields. For example, $(\mathbb{R}, <, +, -, \cdot, 0, 1) \models \mathrm{RCF}$. Similar to the fact that $\mathrm{ACF}_{\forall}$ is the theory of integral domains, $\mathrm{RCF}_{\forall}$ is the theory of ordered integral domains. However, $\mathrm{RCF}$ eliminants quantifiers in $\mathcal{L}_{\mathrm{or}}$ but does not in $\mathcal{L}_{\mathrm{ring}}$. In the latter, $\mathbb{R}_{>0}$ is defined only by an existential formula $\exists y \; x = y^2$. Definable subsets coincide in the languages of rings and ordered rings. The class of real closed fields is an elementary class. In fact, Macintyre proved that the only infinite field with quantifier elimination in the languages of rings is algebraically closed.

# RCF II

### Theorem (Uniqueness)

*Let $F$ be an ordered field. The real closure $K \supseteq F$ is an algebraic extension and the given ordering on $F$ extends to an ordering on $K$. Moreover, if $R_1$ and $R_2$ are two real closures of $F$ with respect to the same ordering on $F$, then there is a unique $F$-isomorphism $R_1 \cong R_2$ preserving the order.*

### Corollary (Algebraically prime)

$\mathrm{RCF}$ *has algebraically prime models.*

### Definition (Semialgebraic sets)

Let $R \models \mathrm{RCF}$. A subset $X \subseteq R^n$ is called semialgebraic if it is defined by a Boolean combination of polynomial equations and inequalities $f(\bar{x}) = 0, g(\bar{x}) > 0, f, g \in R[X_1, \ldots, X_n]$.

# RCF III

### Corollary (Definable sets)

*Let $R \models \mathrm{RCF}$. The definable subsets of $R^n$ are exactly the semialgebraic subsets of $R^n$.*

### Theorem (Tarski-Seidenberg)

*The projection of a semialgebraic set is semialgebraic.*

Proof. Let $R \models \mathrm{RCF}$, and let $X \subseteq R^{n+1}$ be definable. Then $X$ is defined by a quantifier-free formula $\varphi(\bar{x}, y)$. Its projection to $R^n$ is defined by $\exists y \varphi(\bar{x}, y)$. By quantifier elimination, this formula is equivalent modulo $\mathrm{RCF}$ to a quantifier-free formula. Therefore the projection is again semialgebraic. $\square$

# RCF IV

**Theorem (Real Nullstellensatz)**

*Let $R \models \mathrm{RCF}$, and let $I \subseteq R[X_1, \ldots, X_n]$ be an ideal. Let*

$$V_R(I) = \{\bar{a} \in R^n : f(\bar{a}) = 0 \textit{ for all } f \in I\}.$$

*Then $I(V_R(I)) = \sqrt[\mathrm{r}]{I}$, where $\sqrt[\mathrm{r}]{I}$ is the real radical of $I$. Equivalently, $f \in I(V_R(I))$ if and only if there exist $m \geq 1$ and polynomials $g_1, \ldots, g_s$ such that $f^{2m} + g_1^2 + \cdots + g_s^2 \in I$.*

**Theorem (Hilbert's 17th problem)**

*Let $R \models \mathrm{RCF}$, and let $f \in R(X_1, \ldots, X_n)$. If $f(\bar{a}) \geq 0$ wherever $f$ is defined, then $f$ is a sum of squares of rational functions $f = q_1^2 + \cdots + q_m^2, q_i \in R(X_1, \ldots, X_n)$.*

In $R \models \mathrm{RCF}$, a cell is either a point $\{a\}$ or an interval

$$(a, b), \quad (a, \infty), \quad (-\infty, b), \quad R.$$

# RCF V

Suppose cells have been defined in $R^n$. If $C \subseteq R^n$ is a cell and $f, g : C \to R$ are continuous semialgebraic functions with $f < g$, then the following are cells in $R^{n+1}$:

$$\Gamma(f) = \{(\bar{x}, y) : \bar{x} \in C,\ y = f(\bar{x})\},$$

and

$$(f, g)_C = \{(\bar{x}, y) : \bar{x} \in C,\ f(\bar{x}) < y < g(\bar{x})\}.$$

We also allow $f = -\infty$ and $g = +\infty$.

## Theorem (Cell decomposition)

*Every semialgebraic set $X \subseteq R^n$ can be partitioned into finitely many semialgebraic cells.*

## Corollary (O-minimality)

$\mathrm{RCF}$ *is o-minimal.*

# RCF VI

### Theorem (Elimination of imaginaries)

$\mathrm{RCF}$ *eliminates imaginaries.*

Proof. By cell decomposition, real closed fields have definable choice. Now let $E(\bar{x}, \bar{y})$ be a definable equivalence relation on a definable set $X \subseteq R^n$. Apply definable choice to the set $\{(\bar{a}, \bar{b}) \in X \times X : \bar{a} E \bar{b}\}$. We get a definable function $c : X \to X$ such that $c(\bar{a}) \in [\bar{a}]_E$. Choosing the representative uniformly, we may arrange that $\bar{a} E \bar{b} \Leftrightarrow c(\bar{a}) = c(\bar{b})$. Thus each imaginary class $[\bar{a}]_E$ is interdefinable with the real tuple $c(\bar{a})$. Hence $\mathrm{RCF}$ eliminates imaginaries. $\square$

### Theorem (Instability)

$\mathrm{RCF}$ *has the order property and hence is not stable.*

# DCF I

## Example (Differential fields)

Let $\mathcal{L}_\delta = \{+, -, \cdot, 0, 1, \delta\}$. A differential field is a field $K$ together with a map $\delta : K \to K$ such that

$$\delta(x + y) = \delta(x) + \delta(y) \wedge \delta(xy) = x\delta(y) + y\delta(x).$$

The map $\delta$ is called a derivation. More interestingly, $(\mathbb{C}(t), \frac{d}{dt})$ is a differential field. Let $K \models \mathrm{DCF}$. The ring of differential polynomials in one variable over $K$ is $K\{X\} = K[X, \delta X, \delta^2 X, \ldots]$. If $f \in K\{X\}$, the order of $f$, denoted $\mathrm{ord}(f)$, is the largest $n$ such that $\delta^n X$ appears effectively in $f$. A differential field $K$ of characteristic 0 is called differentially closed if every finite system of differential polynomial equations and inequations over $K$ which has a solution in some differential field extension of $K$ already has a solution in $K$. The theory $\mathrm{DCF}_0$ of differentially closed fields of characteristic 0 is complete, eliminates quantifiers and imaginaries,is $\omega$-stable but is not uncountably categorical.

# DCF II

**Theorem (The field of constants)**

*Let $K \models \mathrm{DCF}_0$. Define $C_K = \{x \in K : \delta(x) = 0\}$. Then $C_K$ is an algebraically closed field.*

Proof. It is clear that $C_K$ is a subfield of $K$. We show that it is algebraically closed. Let $f(X) \in C_K[X]$ be a nonconstant polynomial. Consider the finite system $f(x) = 0, \delta(x) = 0$. This system has a solution in a differential field extension of $K$: take an algebraic extension of the constant field $C_K$ containing a root of $f$, extend the derivation by making this root constant, and then form a differential field extension of $K$. Since $K \models \mathrm{DCF}_0$, the same system has a solution $a \in K$. Then $f(a) = 0$ and $\delta(a) = 0$. Thus $a \in C_K$. Hence every nonconstant polynomial over $C_K$ has a root in $C_K$, so $C_K$ is algebraically closed. $\square$

**Theorem (Robinson-Blum)**

$\mathrm{DCF}_0$ *eliminates quantifiers.*

# DCF III

### Theorem (Ritt-Raudenbush)

$\mathrm{DCF}_0$ *eliminates imaginaries.*

### Corollary (Completeness)

*The theory* $\mathrm{DCF}_0$ *is complete.*

Proof. By quantifier elimination, every sentence is equivalent modulo $\mathrm{DCF}_0$ to a quantifier-free sentence. A quantifier-free sentence in $\mathcal{L}_\delta$ has no free variables, so it is a Boolean combination of equalities between differential terms built from 0 and 1. Since $\delta(0) = \delta(1) = 0$, its truth is determined by the field axioms and the condition $\mathrm{char}(K) = 0$. Thus any two models of $\mathrm{DCF}_0$ satisfy the same sentences. Hence $\mathrm{DCF}_0$ is complete. $\square$

### Corollary ($\omega$-stability)

$\mathrm{DCF}_0$ *is* $\omega$*-stable. In particular,* $\mathrm{DCF}_0$ *is stable.*

# Rexp I

**Example (The real exponential field)**

Let $\mathcal{L}_{\exp} = \{<, +, -, \cdot, 0, 1, \exp\}$. The real exponential field is the structure $\mathbb{R}_{\exp} = (\mathbb{R}; <, +, -, \cdot, 0, 1, \exp)$, where $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ is the usual exponential function. For example, the following set

$$\{x \in \mathbb{R} : \exp(x) > x^2 + 1\}$$

and function

$$x \mapsto \exp(x) + x^3$$

are both definable.

**Theorem (Wilkie)**

*The theory $T_{\exp} = \mathrm{Th}(\mathbb{R}_{\exp})$ is model-complete and o-minimal.*

**Corollary (Completeness)**

*The theory $T_{\exp}$ is complete.*

# Rexp II

### Theorem (Elimination of imaginaries)

*$T_{\exp}$ eliminates imaginaries.*

Proof. Every o-minimal expansion of a real closed field has definable choice. That is, if $X \subseteq M^{m+n}$ is definable and every fiber

$$X_a = \{b \in M^n : (a, b) \in X\}$$

is nonempty, then there is a definable function $f : \pi(X) \to M^n$ such that $f(a) \in X_a$ for every $a \in \pi(X)$. Definable choice gives definable representatives for definable equivalence classes. Indeed, let $E(x, y)$ be a definable equivalence relation on a definable set $X \subseteq M^n$. For each class $[x]_E$, apply definable choice to choose a distinguished representative $c(x) \in [x]_E$. Then $xEy \Leftrightarrow c(x) = c(y)$. Thus the imaginary class $[x]_E$ is interdefinable with the real tuple $c(x)$. Therefore $T_{\exp}$ eliminates imaginaries. $\square$

# Rexp III

### Theorem (Instability)

*The theory $T_{\text{exp}}$ is not stable.*

Proof. The formula $\delta(x; y) = x < y$ has the order property. Indeed, in $\mathbb{R}_{\text{exp}}$, choose $a_i = i$ and $b_j = j + \frac{1}{2}$ for $i, j < \omega$. Then $\mathbb{R}_{\text{exp}} \models \delta(a_i; b_j) \Leftrightarrow i < j + \frac{1}{2} \Leftrightarrow i \leq j$. Thus $\delta(x; y)$ has the order property, so $T_{\text{exp}}$ is not stable. $\square$

### Corollary (Categoricity)

*$T_{\text{exp}}$ is not $\kappa$-categorical for any uncountable $\kappa$.*

Proof. The language $\mathcal{L}_{\text{exp}}$ is countable. If a countable complete theory is categorical in some uncountable cardinal, then it is stable. Since $T_{\text{exp}}$ is not stable, it cannot be uncountably categorical. $\square$

# Index for symbols