RIESZ-MARKOV-KAKUTANI REPRESENTATION THEOREM

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Let $(E, \mathscr{O})$ be a compact Hausdorff space and $\mathscr{B}$ its Borel σ-algebra. The set $C = C(E)$ consists of all continuous functions on E, and is equipped with a complete norm $\|f\| = \sup_{x\in E} |f(x)|$.

Theorem 1. *Every positive continuous linear functional ℓ on the Banach space C can be represented as*

$$\ell(f) = \int_E f(x)\mu(dx), \tag{*}$$

where μ is a Borel measure on E with finite total variation.

Lemma 1. *Suppose $\{E_1, E_2, \ldots, E_n\}$ is an open covering of E. There exist continuous functions $h_1, h_2, \ldots, h_n$ satisfying*

1. *$\forall 1 \le i \le n, \operatorname{supp}(h_i) \subset E_i$;*
2. *$\forall 1 \le i \le n, x \in E, 0 \le h_i(x) \le 1$;*
3. *$\forall x \in E, \sum_{i=1}^n h_i(x) = 1$.*

Proof of Lemma 1. Define $F_1 = E \setminus \cup_{j=2}^n E_j$ which is closed and $F_1 \subset E_1$. Since E is compact Hausdorff, it is also Tychonoff, where open sets separate two disjoint closed sets. Thus, there exists an open set V_1 such that $F_1 \subset V_1 \subset \bar{V_1} \subset E_1$. By construction, $\{V_1, E_2, \ldots, E_n\}$ is an open cover of E. Inductively, we can prove that there exists an open cover $\{V_1, \ldots, V_n\}$ that is also a refinement of $\{E_1, E_2, \ldots, E_n\}$ and $\bar{V_i} \subset E_i$ for each $1 \le i \le n$. For each i, the sets $\bar{V_i}$ and E_i^c are disjoint closed sets. By the Urysohn lemma, there exists a continuous function $g_i : E \to [0, 1]$ such that $g_i(x) = 1$ for $x \in \bar{V_i}$ and $g_i(x) = 0$ for $x \in E_i^c$. Thus, $\operatorname{supp}(g_i) \subset E_i$. Define $G(x) = \sum_{i=1}^n g_i(x)$. Since $\{V_1, \ldots, V_n\}$ covers E, for any $x \in E$, there is at least one index j such that $x \in V_j \subset \bar{V_j}$, meaning $g_j(x) = 1$. Therefore, $G(x) \ge 1$ for all $x \in E$. Define $h_i(x) = g_i(x)/G(x)$, which fulfills (1)-(3). □

Proof of Theorem 1. 1. Define for any open set $O \in \mathscr{O}$,

$$\mu(O) = \sup\{\ell(f) : f \in C, 0 \le f \le 1, \operatorname{supp}(f) \subset O\}.$$

By the Urysohn lemma, this set on the RHS is non-empty. We need to show that
1.1. $\mu(\cup_n O_n) = \lim_n \mu(\sum_{m=1}^n O_m)$ for $O_n \in \mathscr{O}$.
For any $f \in C$ with $0 \le f \le 1$ and $K = \operatorname{supp}(f) \subset O = \cup_{n=1}^\infty O_n$. Since $\{O_n\}$ is an open cover of K, there exists a finite subcover such that $K \subset \cup_{m=1}^{n_0} O_m$ for some integer n_0. Thus, $\ell(f) \le \mu(\cup_{m=1}^{n_0} O_m) \le \lim_{n\to\infty} \mu(\cup_{m=1}^n O_m)$. Taking the supremum over all such f yields $\mu(O) \le \lim_{n\to\infty} \mu(\cup_{m=1}^n O_m)$. The reverse inequality holds trivially by the monotonicity of the supremum.
2. Define for any subset $A \in 2^E$,

$$\mu^*(A) = \inf\{\mu(O) : O \in \mathscr{O}, A \subset O\}.$$

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We need to show that
2.1. $\mu^*(\cup_n A_n) \leq \sum_n \mu^*(A_n)$ for $A_n \in 2^E$.
If $\sum_{n=1}^{\infty} \mu^*(A_n) = \infty$, the inequality is trivial. Assume the sum is finite. For any $\varepsilon > 0$ and each n, there exists $O_n \in \mathscr{O}$ with $A_n \subset O_n$ such that $\mu(O_n) < \mu^*(A_n) + \varepsilon/2^n$. Denote $O = \cup_{n=1}^{\infty} O_n \in \mathscr{O}$. Then $A = \cup_{n=1}^{\infty} A_n \subset O$, and by Step 1.1 and finite subadditivity on open sets, $\mu^*(A) \leq \mu(O) \leq \sum_{n=1}^{\infty} \mu(O_n) < \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon$. Since ε is arbitrary, the countable subadditivity follows.
3. A subset $B \in 2^E$ is said to satisfy the Carathéodory condition if $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$ for all $A \in 2^E$. We need to show
3.1. All subsets satisfying Carathéodory condition are indeed a σ-algebra $\bar{\mathscr{B}}$ that contains $\mathscr{O}$.
3.2. $\mu^*|_{\bar{\mathscr{B}}}$ is a measure.
To show $\mathscr{O} \subset \bar{\mathscr{B}}$, it suffices to show that any closed set $F \subset E$ is μ^*-measurable. Since subadditivity already provides $\mu^*(A) \leq \mu^*(A \cap F) + \mu^*(A \backslash F)$, we only need the reverse inequality. Fix $A \in 2^E$ and $\varepsilon > 0$. By the definition of μ^*, there exists an open set $O \supset A$ such that $\mu(O) < \mu^*(A) + \varepsilon$. Since $A \cap F \subset O \cap F$ and $A \backslash F \subset O \backslash F$, it is sufficient to show $\mu(O) \geq \mu^*(O \cap F) + \mu(O \backslash F)$. Since $O \backslash F$ is open, there exists $g_2 \in C$ with $0 \leq g_2 \leq 1$, $\text{supp}(g_2) \subset O \backslash F$, and $\ell(g_2) > \mu(O \backslash F) - \varepsilon$. Let $U = O \backslash \text{supp}(g_2)$. Then U is an open set containing $O \cap F$. By the definition of $\mu(U)$, there exists $g_1 \in C$ with $0 \leq g_1 \leq 1$, $\text{supp}(g_1) \subset U$, and $\ell(g_1) > \mu(U) - \varepsilon \geq \mu^*(O \cap F) - \varepsilon$. Since $\text{supp}(g_1) \cap \text{supp}(g_2) = \emptyset$, we have $0 \leq g_1 + g_2 \leq 1$ and $\text{supp}(g_1 + g_2) \subset O$. By the linearity of ℓ and the definition of $\mu(O)$,

$$\mu(O) \geq \ell(g_1 + g_2) = \ell(g_1) + \ell(g_2) > \mu^*(O \cap F) + \mu(O \backslash F) - 2\varepsilon.$$

As $\varepsilon \to 0$, we obtain $\mu^*(A) \geq \mu^*(A \cap F) + \mu^*(A \backslash F)$. Thus $F \in \bar{\mathscr{B}}$, which implies $\mathscr{O} \subset \bar{\mathscr{B}}$. By Carathéodory's Extension Theorem, $\bar{\mathscr{B}}$ is a σ-algebra containing $\mathscr{B}$, and $\mu^*|_{\bar{\mathscr{B}}}$ is a complete measure.
4. By Lemma 1, we will show (*) is valid.
Assume without loss of generality that $f \geq 0$ with its range in $[0, M]$. For any $\delta > 0$, partition $[0, M]$ with $0 = y_0 < y_1 < \cdots < y_m = M$ such that $y_i - y_{i-1} < \delta$. Define $E_i = \{x \in E : y_{i-1} - \delta < f(x) < y_i + \delta\} \in \mathscr{O}$. By Lemma 1, there exists a partition of unity $\{h_1, \ldots, h_m\}$ subordinate to $\{E_i\}$. Since $f = \sum_{i=1}^m f h_i$ and $(y_{i-1} - \delta) h_i(x) \leq f(x) h_i(x) \leq (y_i + \delta) h_i(x)$, applying ℓ yields

$$\sum_{i=1}^{m} (y_{i-1} - \delta) \ell(h_i) \leq \ell(f) \leq \sum_{i=1}^{m} (y_i + \delta) \ell(h_i).$$

Integrating with respect to μ yields identical bounds

$$\sum_{i=1}^{m} (y_{i-1} - \delta) \int_E h_i(x) \mu(dx) \leq \int_E f(x) \mu(dx) \leq \sum_{i=1}^{m} (y_i + \delta) \int_E h_i(x) \mu(dx).$$

By the construction of μ on open sets, $\ell(h_i) = \int_E h_i(x) \mu(dx)$. Because $y_i - y_{i-1} < \delta$, the upper and lower bounds are within $2\delta \cdot \mu(E)$ of each other. Since $\delta > 0$ is arbitrary, we conclude that $\ell(f) = \int_E f(x) \mu(dx)$. □

Theorem 1 can be extended to any continuous linear functional on C, possibly non-positive due to the Banach lattice property of C. Let $C^+ = \{f \in C : f \geq 0\}$, a positive cone satisfying $c_1 f_1 + c_2 f_2 \in C^+$ for all $c_1, c_2 \geq 0$ and $f_1, f_2 \in C^+$. It's easy to observe that $C = C^+ - C^+$.

Theorem 2. *Every continuous linear functional on C has a representation (*), where μ is a signed Borel measure on E with finite total variation.*

Proof. 1. Define for each $f \in C^+$,

$$\ell^+(f) = \sup\{\ell(g) : 0 \le g \le f\}.$$

Clearly, $\ell^+(f) \ge \ell(0) = 0$. Since ℓ is continuous, $|\ell(g)| \le \|\ell\|\|g\| \le \|\ell\|\|f\|$, which implies $\ell^+(f)$ is finite. We will show that
1.1. ℓ^+ is additive on C^+.
1.2. ℓ^+ uniquely extends to a linear functional on C.
To prove 1.1, let $f_1, f_2 \in C^+$. For any $0 \le g_1 \le f_1$ and $0 \le g_2 \le f_2$, we have $0 \le g_1 + g_2 \le f_1 + f_2$. Thus $\ell(g_1) + \ell(g_2) = \ell(g_1 + g_2) \le \ell^+(f_1 + f_2)$. Taking the supremum over g_1 and g_2, we get $\ell^+(f_1) + \ell^+(f_2) \le \ell^+(f_1 + f_2)$. - Conversely, for any $0 \le g \le f_1 + f_2$, define $g_1 = \min(g, f_1)$ and $g_2 = g - g_1$. It is easy to verify that $0 \le g_1 \le f_1$ and $0 \le g_2 \le f_2$. Since $\ell(g) = \ell(g_1) + \ell(g_2) \le \ell^+(f_1) + \ell^+(f_2)$, taking the supremum over g yields $\ell^+(f_1 + f_2) \le \ell^+(f_1) + \ell^+(f_2)$. To prove 1.2, define $\ell^+(f) = \ell^+(f^+) - \ell^+(f^-)$, where $f^+ = \max(f, 0), f^- = \max(-f, 0)$. Suppose there are two different decompositions $f = f_1 - f_2 = g_1 - g_2$ with $f_i, g_i \in C^+, i = 1, 2$. Then $f_1 + g_2 = g_1 + f_2$. Since $f_1, g_2, g_1, f_2 \in C^+$ and ℓ^+ is additive on C^+, we have

$$\ell^+(f_1) + \ell^+(g_2) = \ell^+(g_1) + \ell^+(f_2) \implies \ell^+(f_1) - \ell^+(f_2) = \ell^+(g_1) - \ell^+(g_2).$$

Thus, the definition of $\ell^+(f)$ is independent of the choice of decomposition. Let $f, g \in C$ and c be any real number. For $f + g$, choosing decompositions $f = f_1 - f_2$ and $g = g_1 - g_2$, we have $f + g = (f_1 + g_1) - (f_2 + g_2)$. The additivity follows directly from the additivity of ℓ^+ on C^+. For cf, if $c \ge 0$, then $cf = cf_1 - cf_2$ and $cf_i \in C^+$. If $c < 0$, then $cf = (-c)f_2 - (-c)f_1$ and $(-c)f_i \in C^+$. In both cases, the homogeneity of ℓ^+ on C^+ ensures $\ell^+(cf) = c\ell^+(f)$.
2. By construction, ℓ^+ is a positive linear functional. Define $\ell^- = \ell^+ - \ell$. For any $f \in C^+$, we have $\ell^+(f) \ge \ell(f)$ by definition. Hence, $\ell^-(f) = \ell^+(f) - \ell(f) \ge 0$, which is also a positive linear functional.
3. Repeat the process conducted in Theorem 1. □

Also, for a locally compact, second-countable Hausdorff space E', we know that it is paracompact, which means any open covering has a locally finite refinement that is also an open covering. In such a space, we have a similar proposition. Let's denote all bounded continuous functions on E' as $C' = C_b(E')$.

Theorem 3. *Every continuous linear functional on C' has a representation (*), where μ is a locally finite signed Borel measure on E with finite total variation.*

Lemma 2. *Suppose $\{E_\alpha\}_{\alpha \in I}$ is an open covering of some topological space E. E is paracompact if and only if there exist continuous functions $\{h_\alpha\}_{\alpha \in I}$ satisfying*

(1) $\forall \alpha \in I, \mathrm{supp}(h_\alpha) \subset E_\alpha$;
(2) $\forall \alpha \in I, x \in E, 0 \le h_\alpha(x) \le 1$;
(3) $\forall x \in E$, *there exists finite index set* $I_0(x) = \{\alpha \in I : h_\alpha(x) \ne 0\}$ *such that* $\sum_{\alpha \in I_0(x)} h_\alpha(x) = 1$.

Based on Lemma 2, Theorem 3 can be seen as a combination of Theorem 1 and 2, whose proof is the same in essence.

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