

A NOTE ON MATRIX JACOBIANS

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We collect some useful matrix transformations of coordinates and derive their Jacobians directly from linear algebra and exterior differentiation.

1. CONVENTIONS AND THE ELEMENTARY LINEAR TRANSFORMATIONS

The scalar polar substitution $x = ru$ in $\mathbb{R}^n$ has volume element $r^{n-1}drd\omega(u)$. The matrix decompositions below are its analogues. In the real case write

$$M_{n,p} = M_{n,p}(\mathbb{R}), \quad \mathcal{S}_p = \{S \in M_{p,p} : S^\top = S\}, \quad \mathcal{P}_p = \{S \in \mathcal{S}_p : S > 0\}.$$

Lebesgue measure in the independent real entries is denoted by

$$dX = \prod_{i=1}^n \prod_{j=1}^p dx_{ij}, \quad dS = \prod_{1 \leq i \leq j \leq p} ds_{ij}.$$

Equality of differential volume elements always refers to absolute Jacobians. We write $F \asymp G$ for an equality of volume elements up to a strictly positive constant independent of the displayed coordinates. Unless explicitly normalized, angular measures are the invariant volume forms induced by the decompositions. Their precise normalization is discussed in Section 3.

Theorem 1.1 (Linear and congruence Jacobians). *Let $A \in \mathrm{GL}_n(\mathbb{R})$ and $B \in \mathrm{GL}_p(\mathbb{R})$. Then*

$$\begin{aligned} (1) \quad & d(AXB) = |\det A|^p |\det B|^n dX, & X \in M_{n,p}, \\ (2) \quad & d(ASA^\top) = |\det A|^{p+1} dS, & S \in \mathcal{S}_p, \quad A \in \mathrm{GL}_p(\mathbb{R}). \end{aligned}$$

Consequently, the inversion maps on $\mathrm{GL}_p(\mathbb{R})$ and $\mathcal{P}_p$ satisfy

$$(3) \quad d(X^{-1}) = |\det X|^{-2p} dX, \quad d(S^{-1}) = |S|^{-(p+1)} dS.$$

Here and below $|S|$ means $\det S$ for a positive definite matrix.

Proof. Column-wise vectorization gives $\mathrm{vec}(AXB) = (B^\top \otimes A) \mathrm{vec} X$. Since the determinant of this linear map is $(\det A)^p (\det B)^n$, the first formula follows.

For the second formula, consider the representation $L_A : S \mapsto ASA^\top$ on $\mathcal{S}_p$. If $A = \mathrm{diag}(a_1, \dots, a_p)$, the p diagonal coordinates are multiplied by a_i^2 , while s_{ij} , $i < j$, is multiplied by $a_i a_j$. Thus

$$|\det L_A| = \prod_i |a_i|^2 \prod_{i < j} |a_i a_j| = |\det A|^{p+1}.$$

The same identity holds for orthogonal A because L_A is orthogonal on $\mathcal{S}_p$ for the Frobenius inner product $\mathrm{tr}(ST)$. Applying a singular-value decomposition to an arbitrary A proves (2).

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Finally, differentiation of $XX^{-1} = I$ gives $d(X^{-1}) = -X^{-1}(dX)X^{-1}$, and similarly $d(S^{-1}) = -S^{-1}(dS)S^{-1}$. Apply (1) and (2), respectively. $\square$

2. TRIANGULAR COORDINATES AND THE MULTIVARIATE GAMMA INTEGRAL

Every $S \in \mathcal{P}_p$ has a unique Cholesky factorization $S = LL^\top$, where $L = (\ell_{ij})$ is lower triangular and $\ell_{ii} > 0$.

Theorem 2.1 (Cholesky Jacobian). *With $dL = \prod_{1 \leq j \leq i \leq p} d\ell_{ij}$,*

$$(4) \quad dS = 2^p \prod_{i=1}^p \ell_{ii}^{p+1-i} dL.$$

Proof. Write

$$L = \begin{pmatrix} a & 0 \\ v & L_0 \end{pmatrix}, \quad S = \begin{pmatrix} a^2 & av^\top \\ av & vv^\top + L_0 L_0^\top \end{pmatrix},$$

where $a > 0$, $v \in \mathbb{R}^{p-1}$ and L_0 is lower triangular. Perform the changes of coordinates in the order

$$(a, v, L_0) \mapsto (a^2, av, L_0) \mapsto (a^2, av, vv^\top + L_0 L_0^\top).$$

The first Jacobian is $2a \cdot a^{p-1} = 2a^p$; for fixed (a, v) , the second has the Jacobian of the Cholesky map of order $p-1$. Induction, starting from $s_{11} = \ell_{11}^2$, yields (4). $\square$

Define, for $\operatorname{Re} a > (p-1)/2$,

$$(5) \quad \Gamma_p(a) = \int_{\mathcal{P}_p} e^{-\operatorname{tr} S} |S|^{a-(p+1)/2} dS.$$

Corollary 2.2 (Multivariate Gamma product). *For $\operatorname{Re} a > (p-1)/2$,*

$$(6) \quad \Gamma_p(a) = \pi^{p(p-1)/4} \prod_{i=1}^p \Gamma\left(a - \frac{i-1}{2}\right).$$

Proof. Under $S = LL^\top$ one has $|S| = \prod_i \ell_{ii}^2$ and $\operatorname{tr} S = \sum_{i \geq j} \ell_{ij}^2$. By Theorem 2.1, the integrals in the off-diagonal entries give $\sqrt{\pi}$ each, while

$$2 \int_0^\infty e^{-t^2} t^{2a-i} dt = \Gamma\left(a - \frac{i-1}{2}\right).$$

There are $p(p-1)/2$ off-diagonal entries, proving the formula. $\square$

3. QR AND STIEFEL POLAR COORDINATES

The Stiefel manifold of p -frames in $\mathbb{R}^n$ is

$$\operatorname{St}(n, p) = \{H \in M_{n,p} : H^\top H = I_p\}, \quad n \geq p.$$

It is a compact homogeneous space $O(n)/O(n-p)$. Its projection $H \mapsto HH^\top$ has image the Grassmann manifold $\operatorname{Gr}(p, n)$; the fibers of this map are the right $O(p)$ -orbits. In particular,

$$\dim \operatorname{St}(n, p) = np - \frac{p(p+1)}{2} = \frac{p(p-1)}{2} + p(n-p).$$

The two summands are, respectively, rotations within the chosen p -plane and movements of that plane in $\mathbb{R}^n$.

Theorem 3.1 (QR and polar Jacobians). *Let $X \in M_{n,p}$ have full column rank. Its unique QR decomposition with positive diagonal is $X = HT$, where $H \in \text{St}(n, p)$ and T is upper triangular with $t_{ii} > 0$. Let $d\omega(H)$ be the invariant Stiefel volume form obtained from successive sphere coordinates. Then*

$$(7) \quad dX = \left(\prod_{i=1}^p t_{ii}^{n-i} \right) dT d\omega(H).$$

The unique symmetric polar decomposition is

$$X = US^{1/2}, \quad S = X^\top X \in \mathcal{P}_p, \quad U = XS^{-1/2} \in \text{St}(n, p),$$

and its Jacobian is

$$(8) \quad dX = 2^{-p} |S|^{(n-p-1)/2} dS d\omega(U).$$

If $\mu_{n,p}$ denotes the probability-normalized invariant measure on $\text{St}(n, p)$, the same formula reads

$$(9) \quad dX = \frac{\pi^{np/2}}{\Gamma_p(n/2)} |S|^{(n-p-1)/2} dS d\mu_{n,p}(U).$$

Proof. The QR factors are obtained by Gram–Schmidt, column by column:

$$x_j = \sum_{i < j} t_{ij} h_i + t_{jj} h_j, \quad h_j \perp h_1, \dots, h_{j-1}, \quad \|h_j\| = 1.$$

Once the preceding columns are fixed, the first $j-1$ coefficients are ordinary Cartesian coordinates. The remaining vector lies in $\mathbb{R}^{n-j+1}$ and has scalar polar coordinates $t_{jj} > 0$ and $h_j \in S^{n-j}$. Hence the j th column contributes t_{jj}^{n-j} . Multiplying over j proves (7).

Since $S = T^\top T$, the upper-triangular version of Theorem 2.1 gives

$$dS = 2^p \prod_{i=1}^p t_{ii}^{p+1-i} dT.$$

Combining this with (7) yields

$$dX = 2^{-p} \prod_i t_{ii}^{n-p-1} dS d\omega(H) = 2^{-p} |S|^{(n-p-1)/2} dS d\omega(H).$$

For a fixed S , the two angular frames satisfy $U = HTS^{-1/2}$, where $TS^{-1/2} \in O(p)$. Right multiplication by this orthogonal matrix preserves invariant Stiefel volume. Since the change $(S, H) \mapsto (S, U)$ is block triangular, the same expression holds with U .

Finally, successive sphere integration gives

$$(10) \quad \text{vol St}(n, p) = \prod_{j=1}^p \frac{2\pi^{(n-j+1)/2}}{\Gamma((n-j+1)/2)} = \frac{2^p \pi^{np/2}}{\Gamma_p(n/2)}.$$

Divide $d\omega$ by this volume to obtain (9). □

Remark 3.2. The factor 2^{-p} in (8) belongs to the specific unnormalized Stiefel form $d\omega$ used here. If only an invariant probability measure is specified, formula (9), not a constant-free expression, is the appropriate exact statement. This distinction is essential when integrating angular coordinates.

4. ORTHOGONAL DIAGONALIZATION AND TYPE A ROOTS

For a symmetric matrix with distinct eigenvalues, write

$$S = Q\Lambda Q^T, \quad \Lambda = \text{diag}(\lambda_1, \dots, \lambda_p), \quad \lambda_1 > \dots > \lambda_p.$$

The matrix Q is determined only up to a simultaneous change of sign of each eigenvector. Thus the true angular variable belongs to $O(p)/\mathcal{D}_p$, where $\mathcal{D}_p = \{\text{diag}(\varepsilon_1, \dots, \varepsilon_p) : \varepsilon_i \in \{\pm 1\}\}$. Let μ_p be the invariant probability measure on this quotient.

Theorem 4.1 (Spectral Jacobian). *On the open set of real symmetric matrices with simple spectrum,*

$$(11) \quad dS = \frac{\pi^{p^2/2}}{\Gamma_p(p/2)} \prod_{i < j} (\lambda_i - \lambda_j) \prod_{i=1}^p d\lambda_i d\mu_p(Q).$$

Proof. Put $\Omega = Q^T dQ$, so $\Omega^T = -\Omega$. Differentiating the decomposition,

$$Q^T(dS)Q = d\Lambda + [\Omega, \Lambda].$$

The diagonal entries are $d\lambda_i$, and for $i < j$ the (i, j) entry is $(\lambda_j - \lambda_i)\Omega_{ij}$. Since orthogonal conjugation preserves Lebesgue measure on $\mathcal{S}_p$, exterior multiplication of the $p(p+1)/2$ independent differentials yields the local identity

$$dS = \prod_{i < j} (\lambda_i - \lambda_j) \prod_i d\lambda_i d\omega(Q), \quad d\omega(Q) = \bigwedge_{i < j} |(Q^T dQ)_{ij}|.$$

Here $d\omega$ is the invariant volume form on the eigenframe quotient, and all eigenvalues have been ordered. As $\text{St}(p, p) = O(p)$, the volume formula (10) gives $\text{vol } O(p) = 2^p \pi^{p^2/2} / \Gamma_p(p/2)$. Quotienting by the 2^p choices of eigenvector signs gives $\text{vol}(O(p)/\mathcal{D}_p) = \pi^{p^2/2} / \Gamma_p(p/2)$, which proves the normalized statement. $\square$

Remark 4.2. For $p = 2$, choose the eigenvector angle $0 \leq \theta < \pi$. The spectral formula becomes $ds_{11}ds_{12}ds_{22} = (\lambda_1 - \lambda_2)d\lambda_1d\lambda_2d\theta$. Since $d\mu_2 = d\theta/\pi$, the normalization constant in (11) is π , as the theorem predicts.

The positive roots of A_{p-1} are $R_+ = \{e_i - e_j : 1 \leq i < j \leq p\}$. Thus the nonconstant Jacobian factor in (11) is precisely

$$(12) \quad \prod_{\alpha \in R_+} \langle \alpha, \lambda \rangle = \prod_{i < j} (\lambda_i - \lambda_j).$$

It is the elementary Vandermonde determinant, now arising from orthogonal orbit directions rather than from an algebraic identity.

Proposition 4.3 (The invariant volume on the positive cone). *The congruence action of $\text{GL}_p(\mathbb{R})$ on $\mathcal{P}_p$ has stabilizer $O(p)$ at I_p , so that $\mathcal{P}_p \simeq \text{GL}_p(\mathbb{R})/O(p)$. Its invariant volume element, up to normalization, is*

$$(13) \quad d_{\text{inv}}S = |S|^{-(p+1)/2} dS.$$

With ordered logarithmic eigenvalues $\lambda_i = e^{t_i}$ it becomes

$$(14) \quad d_{\text{inv}}S \asymp \prod_{i < j} 2 \sinh\left(\frac{t_i - t_j}{2}\right) \prod_i dt_i d\mu_p(Q).$$

Proof. Under $S \mapsto ASA^\top$, equation (2) multiplies dS by $|\det A|^{p+1}$ and $|S|$ by $|\det A|^2$. Their contributions cancel in (13). For $\lambda_i = e^{t_i}$,

$$e^{t_i} - e^{t_j} = 2e^{(t_i+t_j)/2} \sinh\left(\frac{t_i - t_j}{2}\right).$$

The exponential factors from the $p(p-1)/2$ differences, from $\prod_i d\lambda_i$, and from $|S|^{-(p+1)/2}$ cancel exactly, giving (14). $\square$

5. MATRIX FUNCTIONS AND THEIR JACOBIANS

The preceding spectral formula computes nonlinear transformations of symmetric matrices without differentiating each entry separately.

Theorem 5.1 (Spectral functional calculus). *Let $I \subset \mathbb{R}$ be an interval and $f : I \rightarrow \mathbb{R}$ a C^1 function with nonvanishing derivative, injective on I . For symmetric S with spectrum in I , define $F(S) = Q \operatorname{diag}(f(\lambda_1), \dots, f(\lambda_p)) Q^\top$. Then its absolute Jacobian is*

$$(15) \quad J_F(S) = \prod_{i=1}^p |f'(\lambda_i)| \prod_{i < j} \left| \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} \right|.$$

The divided differences at coincident eigenvalues are interpreted by continuity.

Proof. For a matrix of simple spectrum, compare the spectral volume (11) for S and $F(S)$. The angular frame is unchanged; $df(\lambda_i) = f'(\lambda_i)d\lambda_i$, and each Vandermonde factor changes by the corresponding divided difference. This gives (15). Continuity of the derivative of the matrix functional calculus extends the determinant formula to repeated eigenvalues. $\square$

Corollary 5.2 (Powers, inversion, and fractional maps). *For $R \in \mathcal{P}_p$,*

$$(16) \quad d(R^2) = 2^p |R| \prod_{i < j} (r_i + r_j) dR,$$

$$(17) \quad d(R^{-1}) = |R|^{-(p+1)} dR,$$

$$(18) \quad d(R(I_p + R)^{-1}) = |I_p + R|^{-(p+1)} dR.$$

Here $r_1, \dots, r_p$ are the eigenvalues of R .

Proof. Apply (15) to $f(t) = t^2$, $f(t) = t^{-1}$, and $f(t) = t/(1+t)$, respectively. For instance, $(r_i^2 - r_j^2)/(r_i - r_j) = r_i + r_j$; for $f(t) = t/(1+t)$ the divided difference is $((1+r_i)(1+r_j))^{-1}$, giving the last exponent $p+1$. $\square$

6. SINGULAR VALUES AND TYPES B AND D

Let $X \in M_{n,p}$, $n \geq p$, have rank p and distinct positive singular values $s_1 > \dots > s_p > 0$. Its thin singular-value decomposition is

$$(19) \quad X = H \operatorname{diag}(s_1, \dots, s_p) V^\top, \quad H \in \operatorname{St}(n, p), \quad V \in O(p).$$

The pair (H, V) is unique only up to simultaneous right multiplication by $D \in \mathcal{D}_p$. We denote the corresponding compact angular quotient by $\mathcal{A}_{n,p} = (\operatorname{St}(n, p) \times O(p))/\mathcal{D}_p$, equipped with its invariant probability measure $\rho_{n,p}$.

Theorem 6.1 (Singular-value Jacobian). *For ordered distinct singular values as above,*

$$(20) \quad dX = C_{n,p} \prod_{i=1}^p s_i^{n-p} \prod_{i<j} (s_i^2 - s_j^2) \prod_i ds_i d\rho_{n,p}(H, V),$$

where, with the stated probability normalization,

$$(21) \quad C_{n,p} = \frac{2^p \pi^{p(n+p)/2}}{\Gamma_p(n/2) \Gamma_p(p/2)}.$$

Proof. Apply the polar decomposition $X = US^{1/2}$ from Theorem 3.1, and then diagonalize

$$S = X^\top X = V \operatorname{diag}(s_1^2, \dots, s_p^2) V^\top.$$

The radial part of the polar Jacobian contributes

$$|S|^{(n-p-1)/2} = \prod_i s_i^{n-p-1}.$$

The spectral Jacobian contributes $\prod_{i<j} (s_i^2 - s_j^2)$, and $\prod_i d(s_i^2) = 2^p \prod_i s_i ds_i$. Multiplying these factors proves the powers and differences in (20). The new left frame is $H = UV$; the map $U \mapsto UV$ preserves Stiefel invariant measure for fixed V . The simultaneous sign ambiguity is precisely the quotient defining $\mathcal{A}_{n,p}$. Multiplication of the two normalized constants in (9), (11), and the factor 2^p gives (21). $\square$

For $p = 1$, formula (20) reduces to the familiar polar element on $\mathbb{R}^n$, $dx = |S^{n-1}| s^{n-1} ds d\mu(u)$.

The factor in (20) can be written as

$$(22) \quad \prod_i s_i^{n-p} \prod_{i<j} (s_i - s_j)(s_i + s_j).$$

For $n > p \geq 2$, these are the positive-root factors of type B_p : e_i has multiplicity $n - p$, and $e_i \pm e_j$ has multiplicity one. For $n = p \geq 2$, the e_i factor disappears and one obtains type D_p (with $D_2 \simeq A_1 \times A_1$). These are identities of Jacobian factors; they do not require introducing any probabilistic model.

7. SIMULTANEOUS DIAGONALIZATION AND GENERALIZED EIGENVALUES

The spectral and singular-value transformations are the basic changes A and D in Tumura (1965). Two further transformations there, B and C, concern pairs and triples of matrices. The first has a particularly simple direct Jacobian calculation.

Theorem 7.1 (Congruence diagonalization; Tumura's B). *Let $U \in \mathcal{P}_p$ and $V \in \mathcal{S}_p$. Assume that $U^{-1/2} V U^{-1/2}$ has distinct eigenvalues $\lambda_1 > \dots > \lambda_p$. Then*

$$U = T T^\top, \quad V = T \Lambda T^\top, \quad \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_p)$$

for an invertible T . The local Jacobian is

$$(23) \quad dU dV = 2^p |\det T|^{p+2} \prod_{i<j} (\lambda_i - \lambda_j) dT \prod_i d\lambda_i.$$

This is a local coordinate identity: for fixed ordered Λ , the matrices T and TD , $D \in \mathcal{D}_p$, represent the same pair (U, V) . A global integral over all such T must therefore be divided by 2^p , or restricted to a measurable sign section.

Proof. Existence follows by orthogonally diagonalizing $U^{-1/2}VU^{-1/2} = Q\Lambda Q^\top$ and taking $T = U^{1/2}Q$.

To compute the Jacobian, use the infinitesimal coordinates $E = T^{-1}dT$ and temporarily conjugate both differentials by T^{-1} and $T^{-\top}$. We obtain

$$\begin{aligned} T^{-1}(dU)T^{-\top} &= E + E^\top, \\ T^{-1}(dV)T^{-\top} &= E\Lambda + \Lambda E^\top + d\Lambda. \end{aligned}$$

For each diagonal position i , the map from $(E_{ii}, d\lambda_i)$ to the two diagonal entries has matrix

$$\begin{pmatrix} 2 & 0 \\ 2\lambda_i & 1 \end{pmatrix},$$

whose determinant is 2. For each $i < j$, the map from (E_{ij}, E_{ji}) to the two (i, j) entries has matrix

$$\begin{pmatrix} 1 & 1 \\ \lambda_j & \lambda_i \end{pmatrix},$$

whose determinant is $\lambda_i - \lambda_j$. Their product is $2^p \prod_{i < j} (\lambda_i - \lambda_j)$. Finally, congruence by T acts on each of the two symmetric matrix differentials with determinant $|\det T|^{p+1}$, whereas $dT = TE$ contributes $|\det T|^p$. The net factor is $|\det T|^{2(p+1)-p} = |\det T|^{p+2}$, which proves (23). $\square$

Remark 7.2 (Tumura's C and redundant angular coordinates). For $U \in \mathcal{P}_p$, $V \in \mathcal{P}_q$, and $W \in M_{p,q}$, one may choose invertible matrices A, B so that

$$U = AA^\top, \quad V = BB^\top, \quad W = A \operatorname{diag}_{p \times q}(s_1, \dots, s_p) B^\top \quad (p \leq q).$$

Indeed, whiten W to $U^{-1/2}WV^{-1/2}$ and apply its SVD. For $q > p$, the full right orthogonal factor has an $O(q-p)$ stabilizer. Except in the cases when that stabilizer is discrete, this is not a one-to-one system of coordinates if all entries of A, B are treated as independent. One should use the corresponding angular quotient or an integral formula. Tumura's treatment of rank-deficient transformations makes this point explicit.

8. THE MATRIX BETA SUBSTITUTION

Let $A, B \in \mathcal{P}_p$, and put

$$(24) \quad T = A + B, \quad Y = T^{-1/2}AT^{-1/2}.$$

Then $T \in \mathcal{P}_p$ and $0 < Y < I_p$, and the inverse map is $A = T^{1/2}YT^{1/2}$, $B = T^{1/2}(I_p - Y)T^{1/2}$.

Theorem 8.1 (Matrix Beta Jacobian). *The map (24) is a smooth bijection and*

$$(25) \quad d\operatorname{Ad}B = |T|^{(p+1)/2} dT dY.$$

Proof. The linear change $(A, B) \mapsto (T = A + B, A)$ has determinant one on $\mathcal{S}_p \oplus \mathcal{S}_p$. Next, with T held fixed, the map $Y \mapsto A = T^{1/2}YT^{1/2}$ has, by (2), Jacobian $|\det T^{1/2}|^{p+1} = |T|^{(p+1)/2}$. Terms arising from differentiating $T^{1/2}$ affect only the T -columns of the full derivative, which is block triangular in (T, Y) . They do not change its determinant. $\square$

For $\operatorname{Re} a, \operatorname{Re} b > (p-1)/2$, define

$$B_p(a, b) = \int_{0 < Y < I_p} |Y|^{a-(p+1)/2} |I_p - Y|^{b-(p+1)/2} dY.$$

Corollary 8.2 (Gamma–Beta identity and the second Beta integral). *One has*

$$(26) \quad B_p(a, b) = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}.$$

Equivalently,

$$(27) \quad B_p(a, b) = \int_{\mathcal{P}_p} |R|^{a-(p+1)/2} |I_p + R|^{-(a+b)} dR.$$

Proof. By Tonelli’s theorem, or by absolute convergence for complex parameters,

$$\Gamma_p(a)\Gamma_p(b) = \int_{\mathcal{P}_p \times \mathcal{P}_p} e^{-\operatorname{tr}(A+B)} |A|^{a-(p+1)/2} |B|^{b-(p+1)/2} dA dB.$$

The substitution in Theorem 8.1 separates the integrand into $e^{-\operatorname{tr} T} |T|^{a+b-(p+1)/2}$ and $|Y|^{a-(p+1)/2} |I_p - Y|^{b-(p+1)/2}$, proving (26). Finally, set $Y = R(I_p + R)^{-1}$. Then $|Y| = |R|/|I_p + R|$ and $|I_p - Y| = |I_p + R|^{-1}$; use (18) to obtain (27). $\square$

9. THE CLASSICAL FIELDS AND ROOT MULTIPLICITIES

The factorization into roots is not restricted to real matrices. Put $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ and write $d = \dim_{\mathbb{R}} \mathbb{F} = 1, 2$, or 4 . For $\mathbb{F} = \mathbb{H}$, matrices are interpreted as quaternionic matrices, X^* denotes conjugate transpose, and $|\det_{\mathbb{F}} A|$ denotes the product of the singular values of A ; Hermitian matrices have real eigenvalues. All volume elements are Lebesgue measures on the underlying real vector spaces.

Proposition 9.1 (Jacobians over the three classical fields). *For a positive definite Hermitian $p \times p$ matrix S over $\mathbb{F}$, a nonsingular A , and a lower triangular Cholesky factor L with positive real diagonal,*

$$(28) \quad d(ASA^*) = |\det_{\mathbb{F}} A|^{d(p-1)+2} dS,$$

$$(29) \quad d(LL^*) = 2^p \prod_{i=1}^p \ell_{ii}^{d(p-i)+1} dL.$$

With invariant angular measures, the spectral Jacobian is

$$(30) \quad dS \asymp \prod_{i < j} |\lambda_i - \lambda_j|^d \prod_i d\lambda_i d\mu(Q).$$

If $X \in M_{n,p}(\mathbb{F})$, $n \geq p$, has distinct positive singular values $s_1, \dots, s_p$, then

$$(31) \quad dX \asymp \prod_i s_i^{d(n-p+1)-1} \prod_{i < j} |s_i^2 - s_j^2|^d \prod_i ds_i d\rho.$$

Proof. By unitary invariance it suffices to calculate the congruence Jacobian for a diagonal $A = \operatorname{diag}(a_i)$ with $a_i > 0$. Every diagonal Hermitian coordinate is real and scales as a_i^2 ; every off-diagonal coordinate lies in the real d -dimensional space $\mathbb{F}$ and scales as $a_i a_j$. The exponent of each a_i is therefore $2 + d(p-1)$, proving (28). The inductive block argument for Theorem 2.1 has one real diagonal coordinate and $p-i$ off-diagonal coordinates of real dimension d in row/column i , giving (29).

In the spectral calculation, $Q^*(dS)Q = d\Lambda + [Q^*dQ, \Lambda]$. For each $i < j$, the independent off-diagonal entry now has d real coordinates, all multiplied by $\lambda_j - \lambda_i$. This proves (30).

The Gram–Schmidt proof of (7) over $\mathbb{F}$ has radial factor $t_{ii}^{d(n-i+1)-1}$ at column i . Dividing by (29) for $S = X^*X = T^*T$ yields the polar factor $|S|^{d(n-p+1)/2-1}$. Diagonalize S , set $\lambda_i = s_i^2$, and use (30); the powers of s_i become $2(d(n-p+1)/2-1) + 1 = d(n-p+1) - 1$, yielding (31). $\square$

For the positive-definite Hermitian cone, the root factors are those of A_{p-1} with multiplicity d . For rectangular singular values, the factor (31) can be written

$$\prod_{i < j} |s_i - s_j|^d |s_i + s_j|^d \prod_i s_i^{d(n-p)} s_i^{d-1}.$$

These are root factors for $e_i - e_j$, $e_i + e_j$, e_i , and $2e_i$, respectively, with multiplicities

$$(32) \quad m_{e_i \pm e_j} = d, \quad m_{e_i} = d(n-p), \quad m_{2e_i} = d-1.$$

The harmless constant 2^{d-1} contributed by each root $2e_i$ is absorbed into the angular normalization. For $p \geq 2$ the resulting reduced or nonreduced root types are as follows:

Field	$n = p$	$n > p$
$\mathbb{R} \ (d = 1)$	D_p	B_p
$\mathbb{C}, \mathbb{H} \ (d = 2, 4)$	C_p	BC_p

This is a statement about root systems and their geometric multiplicities. In the theory of symmetric functions, the same multiplicity d corresponds to the Jack parameter $\alpha = 2/d$.

10. RADIAL DIFFERENTIAL OPERATORS

One benefit of the root interpretation is that the Jacobian immediately produces the radial part of the Euclidean Laplacian. For any open Weyl chamber with a positive Jacobian $J(x)$, the operator

$$(33) \quad \mathcal{L}_J f = \frac{1}{J(x)} \sum_i \frac{\partial}{\partial x_i} \left(J(x) \frac{\partial f}{\partial x_i} \right)$$

is symmetric with respect to $J(x)dx$ on compactly supported smooth functions.

Proposition 10.1 (Radial Laplacians). *For a conjugation-invariant function on S_p depending only on the eigenvalues λ_i , the radial part of the Euclidean Laplacian (for the Frobenius metric) is*

$$(34) \quad \sum_{i=1}^p \partial_{\lambda_i}^2 + \sum_{i < j} \frac{\partial_{\lambda_i} - \partial_{\lambda_j}}{\lambda_i - \lambda_j}.$$

For a left-and-right orthogonally invariant function on $M_{n,p}$ depending only on the distinct singular values $s_i > 0$, it is

$$(35) \quad \sum_i \partial_{s_i}^2 + (n-p) \sum_i \frac{\partial_{s_i}}{s_i} + \sum_{i < j} \left(\frac{\partial_{s_i} - \partial_{s_j}}{s_i - s_j} + \frac{\partial_{s_i} + \partial_{s_j}}{s_i + s_j} \right).$$

For $\mathbb{F}$ of real dimension d , multiply the pair-interaction terms in both formulas by d and replace the coefficient $n-p$ in (35) by $d(n-p+1) - 1$.

Proof. Both the orthogonal conjugation orbits and the left-right orthogonal orbits meet the diagonal slices orthogonally for the Frobenius metric. Along these slices the coordinates λ_i and s_i have the Euclidean metric $\sum_i d\lambda_i^2$ and $\sum_i ds_i^2$. The radial Laplace–Beltrami formula is therefore (33), with $J_A(\lambda) = \prod_{i < j} (\lambda_i - \lambda_j)$ or $J_{B/D}(s) = \prod_i s_i^{n-p} \prod_{i < j} (s_i^2 - s_j^2)$. Logarithmic differentiation gives

$$\begin{aligned}\partial_{\lambda_i} \log J_A &= \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \\ \partial_{s_i} \log J_{B/D} &= \frac{n-p}{s_i} + \sum_{j \neq i} \left(\frac{1}{s_i - s_j} + \frac{1}{s_i + s_j} \right).\end{aligned}$$

Substitution into (33), followed by pairing the i, j terms, gives the two formulas. Replacing the powers by those in (30) and (31) proves the field extension. $\square$

Remark 10.2. The formulas are expressed on the open set of distinct eigenvalues or singular values. Their apparent singularities are removable when acting on smooth invariant functions. The coefficients are precisely those that occur in the radial Dunkl-type operators for the roots A , B , C , D , and BC ; this is the link between matrix-coordinate changes and the special-function theory that follows.